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Eigenvalues and Eigenvectors of an m-Polar Fuzzy Matrix

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ABSTRACT

In fuzzy linear algebra, the concept of eigenvalues and eigenvectors (E. Values and E. Vectors) plays a vital role. In order to build up the linear space we set up in this paper, the similarity relations, E. Values and E. Vectors of m-polar fuzzy matrices (mPFMs). Here, we discussed idempotent, row and column diagonally dominant and spectral radius of mPFMs. In addition, a few properties and results of E. Values and E. Vectors of mPFMs are proved.

Keywords: *Relation, m-polar fuzzy matrix, vector, eigenvalue, eigenvector, spectral radius.*

1. Introduction

Fuzzy sets were developed using continuous parameters to solve problems related to vague and uncertain real life situations were demonstrated by Zadeh [9] in 1965. Problems related to networks that demand intuitive data analysis technique were solved by interval valued fuzzy sets introduced by Zadeh [10]. The limitations of traditional model were overcome by the introduction of bipolar fuzzy concept in 1994 by Zhang [11, 12]. This was further improved by Chen et al. [2] to m-polar fuzzy set (mPFS).

Related to fuzzy matrices, a lot of works are accessible for E. Values and E. Vectors [1, 3, 4]. Although, their procedures were not appropriate for all types of matrices and these are extremely difficult methods. Really, it is hard job to compute E. Values and E. Vectors for a fuzzy matrix. A few researchers tried to compute out the E. Values and E. Vectors to script matrices as per rules of introducing α-cut method [7, 8]. Using max-min and min-max operations Mondal and Pal [6] found the E. Values and E. Vectors to the bipolar fuzzy matrices. But, in fuzzy concept m-values are suitable. In viewing this state in mind we are flexible to compute out that E. Values and E. Vectors those m-values and lies in [0, 1].

In this paper, we have used the max-min operation in the equation $QX = \lambda X$ or $XQ = \lambda X$ to compute λ and X . This is reasonable and usual in fuzzy situation. This is the first endeavor to compute λ and X by means of max-min operation to mPFMs.

2. Preliminaries

An m-polar fuzzy set (mPFS) is most familiar and extension of fuzzy set with more than two membership values. In this section, a few fundamental notions of mPFS are introduced. Also some necessary binary operations like $+, \cdot, \times$ on mPFSs are specified.

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Definition 1. An *m-polar fuzzy set* [mPFS] M_F in X is an object of the form $M_F = \{(s, \psi_1(s), \psi_2(s), \cdots, \psi_m(s))\}$ where $\psi_1, \psi_2, \cdots, \psi_m : X \to [0,1]$ are m functions.

Definition 2. Let $\alpha, \beta \in M_F$, where $\alpha = \langle \alpha_1, \alpha_2, \cdots, \alpha_m \rangle$ and $\beta = \langle \beta_1, \beta_2, \cdots, \beta_m \rangle$ then the *equality* of α and β can be defined as $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_m = \beta_m$ and it is denoted by $\alpha = \beta$.

Definition 3. Let $\tau, \gamma \in M_F$ where $\tau = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle$, $\gamma = \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle$ and $\tau_1, \tau_2, \cdots, \tau_m$ and $\gamma_1, \gamma_2, \cdots, \gamma_m \in [0, 1]$ then

The *disjunction* of τ and γ is denoted by $\tau + \gamma$ and is given by

$$\tau + \gamma = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle + \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle$$
$$= \langle \max\{\tau_1, \gamma_1\}, \max\{\tau_2, \gamma_2\}, \cdots, \max\{\tau_m, \gamma_m\} \rangle = \langle \tau_1 \vee \gamma_1, \tau_2 \vee \gamma_2, \cdots, \tau_m \vee \gamma_m \rangle.$$

The *parallel conjunction* of τ and γ is denoted by $\tau . \gamma$ and is given by

$$\tau . \gamma = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle . \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle$$
$$= \langle \min\{\tau_1, \gamma_1\}, \min\{\tau_2, \gamma_2\}, \cdots, \min\{\tau_m, \gamma_m\} \rangle = \langle \tau_1 \wedge \gamma_1, \tau_2 \wedge \gamma_2, \cdots, \tau_m \wedge \gamma_m \rangle.$$

Definition 4. Let U_1 and U_2 be two universe of discourses and $X = \{\tau = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle | \tau \in U_1\}$, $Y = \{\gamma = \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle | \gamma \in U_2\}$ be two mPFSs.

The *Cartesian product* of X and Y is given by $X \times Y = \{(\tau, \gamma) \,|\, \tau \in U_1 \text{ and } \gamma \in U_2\}$.

Definition 5. An *m-polar fuzzy relation* (mPFR) between two mPFSs X and Y is defined as a mPFS in $X \times Y$. If R is a relation between X and Y, $\tau \in X$ and $\gamma \in Y$, and if $\psi_1(\tau, \gamma), \psi_2(\tau, \gamma), \cdots, \psi_m(\tau, \gamma)$ are the m membership values to which τ is in relation R with γ, then $\psi = \langle \psi_1, \psi_2, \cdots, \psi_m \rangle \in R$.

Definition 6. Let $\tau, \gamma \in M_F$ where $\tau = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle$, $\gamma = \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle$ then $\tau \le \gamma$ iff $\tau_1 \le \gamma_1, \tau_2 \le \gamma_2, \cdots, \tau_m \le \gamma_m$. i.e., $\tau \le \gamma$ iff $\tau + \gamma = \gamma$.

Definition 7. Let M_F be an mPFS on X and let $\tau, \gamma \in M_F$, where $\tau = \langle \tau_1, \tau_2, \cdots, \tau_m \rangle$, $\gamma = \langle \gamma_1, \gamma_2, \cdots, \gamma_m \rangle$, then $\tau < \gamma$ iff $\tau \le \gamma$ and $\tau \ne \gamma$.

Definition 8. An *m-polar fuzzy matrix* $X = [\langle x_{1_{lk}}, x_{2_{lk}}, \cdots, x_{m_{lk}} \rangle]$ is a matrix on fuzzy algebra. The zero matrix O_r is a square matrix of order r in which each elements are $O_m = \langle 0, 0, \cdots, 0 \rangle$ and I_r is an identity matrix of order r whose elements of the diagonal are $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$ and the non-diagonal elements are $O_m = \langle 0, 0, \cdots, 0 \rangle$.

The set M_{rk} is the set of $r \times k$ rectangular mPFMs and M_r, the set of $r \times r$ matrices.

From the definition, we have if $Q = [q_{lk}]_{r \times k} \in M_{rk}$, then $q_{lk} = \langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \rangle \in M_F$, where $q_{lk_1}, q_{lk_2}, \cdots, q_{lk_m} \in [0, 1]$ are the m-membership values of the element q_{lk} respectively.

The operations on mPFMs are as follows:

Definition 9. Let $U = [u_{lk}], V = [v_{lk}] \in M_{th}$ be two mPFMs. Therefore, $u_{lk}, v_{lk} \in M_F$, then $U + V = [u_{lk} + v_{lk}]_{t \times h} = [\langle \max\{u_{1_{lk}}, v_{1_{lk}}\}, \max\{u_{2_{lk}}, v_{2_{lk}}\}, \cdots, \max\{u_{m_{lk}}, v_{m_{lk}}\} \rangle]_{t \times h}$ and

$$U \cdot V = [u_{lk} \cdot v_{lk}]_{t\times h} = \left[\left\langle \min\left\{u_{1_{lk}}, v_{1_{lk}}\right\}, \min\left\{u_{2_{lk}}, v_{2_{lk}}\right\}, \cdots, \min\left\{u_{m_{lk}}, v_{m_{lk}}\right\}\right\rangle\right]_{t\times h}.$$

Definition 10. Let $U = [u_{lk}] \in M_{th}$, $V = [v_{lk}] \in M_{hg}$ be two mPFMs. Therefore, $u_{lk}, v_{lk} \in M_F$, then

$$U \odot V = \left(\sum_{q=1}^{h} u_{lq} \cdot v_{qk}\right)_{t\times g}$$

$$= \left[\left\langle \max_{q=1}^{h}\left(\min\left\{u_{1_{lq}}, v_{1_{qk}}\right\}\right), \max_{q=1}^{h}\left(\min\left\{u_{2_{lq}}, v_{2_{qk}}\right\}\right), \cdots, \max_{q=1}^{h}\left(\min\left\{u_{m_{lq}}, v_{m_{qk}}\right\}\right)\right\rangle\right]_{t\times g}.$$

$$U \otimes V = \left(\prod_{q=1}^{h}\left\{u_{lq} + v_{qk}\right\}\right)_{t\times g}$$

$$= \left(\min_{q=1}^{h}\left[\max\left\{u_{1_{lq}}, v_{1_{qk}}\right\}\right], \min_{q=1}^{h}\left[\max\left\{u_{2_{lq}}, v_{2_{qk}}\right\}\right], \cdots, \min_{q=1}^{h}\left[\max\left\{u_{m_{lq}}, v_{m_{qk}}\right\}\right]\right)_{t\times g}.$$

3. m-Polar fuzzy vector space

The theory of fuzzy vector space was first proposed by Katsaras and Liu [5]. Some elementary concepts of m-polar fuzzy vector space (mPFVS) in terms of mPFA were given below.

Definition 11. An *m-Polar fuzzy vector* (mPFV) is an m-tuple $[v_1, v_2, \cdots, v_m]$ where each element $v_i \in M_F$, $0 \le i \le m$.

Definition 12. An *m-Polar fuzzy vector space* (mPFVS) is an ordered pair $(F, M(v))$, where F is a vector space in crisp sense over the real field R and $M : F \to ([0, 1]^m)$ is the m-polar fuzzy membership mapping with the property that for all $p, q \in$ R and $l, k \in F$, we have $M_1(pl+qk) \ge M_1(l) \wedge M_1(k), M_2(pl+qk) \ge M_2(l) \wedge M_2(k), \cdots, M_m(pl+qk) \ge M_m(l) \wedge M_m(k)$.

Example 13. Let S_m denote the set of all m-tuples $[a_1, a_2, \cdots, a_m]$ over M_F. An element of S_m is called a mPFV of dimension m. For $a = [a_1, a_2, \cdots, a_m]$ and $b = [b_1, b_2, \cdots, b_m]$ in S_m, the following operations addition $(+)$ and multiplication $(\cdot)$ are defined as $a+b = [a_1+b_1, a_2+b_2, \cdots, a_m+b_m] \in S_m$ and for any $l \in M_F$, $la = [la_1, la_2, \cdots, la_m] \in S_m$. The set S_m together with these operations of component wise addition and scalar multiplication is an mPFVS over M_F, as the scalars are restricted in M_F.

Definition 14. Let $S^m = \{u^t | u \in S_m\}$ where u^t the transpose of the vector u. For $a, b \in S^m$ and $l \in M_F$ we define $lb = (lb^t)^t$, $a+b = (a^t + b^t)^t$. Then S^m is an mPFVS. If the order of S_m is $1\times m$, it is a row vector and the element of S^m is called column vectors. Further, $S_m \cong S^m$.

4. Similarity relation on m-polar fuzzy sets

The reflexive, symmetric, transitive relations on mPFMS were established and proved below.

Let $R(X, X)$ be an mPFR on a set X. Let $\psi_1, \psi_2, \cdots, \psi_m : X \times X \to [0,1]$ be the

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membership functions and M_R be an mPFM with respect to R.

Definition 15. If all the diagonal elements of the matrix M_R are $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$, i.e., $\psi_1(l, l) = \psi_2(l, l) = \cdots = \psi_m(l, l) = 1.0$ for all $l \in X$ then $R(X, X)$ is *reflexive.*

Definition 16. If the transpose of M_R is itself i.e., $\psi_1(l, k) = \psi_1(k, l), \psi_2(l, k) = \psi_2(k, l), \cdots,$ $\psi_m(l, k) = \psi_m(k, l)$, for all $l, k \in X$ then $R(X, X)$ is *symmetric.*

Definition 17. If $M_R \geq M_R^2$, i.e., $\psi_1(l, k) \geq \max_{p \in X} \{\min\{\psi_1(l, p), \psi_1(p, k)\}\}$, $\psi_2(l, k) \geq \max_{p \in X} \{\min\{\psi_2(l, p), \psi_2(p, k)\}\}$, $\cdots$, $\psi_m(l, k) \geq \max_{p \in X} \{\min\{\psi_m(l, p), \psi_m(p, k)\}\}$, for all $(l, k) \in X \times X$ then $R(X, X)$ is *transitive.*

Definition 18. The relation $R(X, X)$ is *similarity relation* iff $R(X, X)$ is reflexive, symmetric and transitive.

Proposition 19. *For an mPFM $Q \in M_m$, Q is reflexive if $Q \geq I_m$.*

Proof. Since $Q \geq I_m$, we have all the elements of diagonal of matrix Q are $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$. Therefore the matrix Q is reflexive.

Definition 20. Let $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \right\rangle\right] \in M_m$ be an mPFM. Then we discuss the following mPFSs:

Nature of Q	Condition
Reflexive	$Q \geq I_m$.
Weakly reflexive	$q_{ll} \geq q_{lk}$ for all $0 \leq l, k \leq m$.
Symmetric	$Q = Q^T$.
Idempotent	$Q = Q^2$.
Transitive	$Q^2 \leq Q$.

Proposition 21. *Let $Q \in M_m$ be a reflexive mPFM. Then*

i. Q^T *is reflexive mPFM,*

ii. Q^n *is reflexive mPFM for some* $n \in N$,

iii. $QR \geq R$ *for* $R \in M_m$,

iv. $RQ \geq R$ *for* $R \in M_m$,

v. QR *and* RQ *are reflexive mPFMs if* R *is reflexive,*

vi. QQ^T *and* Q^TQ *are reflexive mPFMs*

Proof. *i)* Since Q is reflexive and all of its diagonal elements are $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$, we have the diagonal entries of Q^T are also $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$. Hence Q^T is reflexive.

ii) Since Q is reflexive, $Q \geq I_m$, we have $Q^2 \geq Q \geq I_m$. Continuing in the same way, we have $Q^n \geq Q^{n-1} \geq \cdots \geq Q^2 \geq Q \geq I_m$ for any $n \in N$. So Q^n is reflexive.

iii) If $Q \geq I_m$ then $QR \geq I_m R \Rightarrow QR \geq R$.

iv) Also $RQ \geq I_m R$ or $RQ \geq R$.

v) As Q is reflexive, $Q \geq I_m$, we have $QR \geq R \geq I_m$ and $RQ \geq R \geq I_m$. So QR and RQ are also reflexive.
vi) Clearly from *i)* and *v),* we have QQ^T and Q^TQ are reflexive.

Proposition 22. *A matrix $Q \in M_m$ is idempotent if it is both transitive and reflexive.*

Proof. As Q is reflexive $Q \geq I_m$, we have $Q^2 \geq Q \geq I_m$. (1)

And Q is transitive implies $Q^2 \leq Q$. (2)

From (1) and (2), $Q^2 = Q$.

Hence Q is idempotent.

The converse of the above proposition is not true as shown in the below example.

Example 23. Let $Q = \begin{bmatrix} \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \\ \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \end{bmatrix}$ not greater than or equals to I_2.

Hence Q is not reflexive. But

$$Q^2 = \begin{bmatrix} \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \\ \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \end{bmatrix} \odot \begin{bmatrix} \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \\ \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \end{bmatrix}$$

$= \begin{bmatrix} \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \\ \langle 0.3,\ 0.5, 0.2\rangle & \langle 0.3,\ 0.5, 0.2\rangle \end{bmatrix} = Q$. i.e., Q is idempotent.

Proposition 24. *If W and Y are two symmetric mPFMs in M_m such that $WY = YW$, then WY is symmetric.*

Proof. It is obvious from the above definitions.

Proposition 25. *If W and Y are two transitive mPFMs in M_m such that $WY = YW$, then WY is transitive.*

Proof. Since W and Y are transitive, $W^2 \leq W$ and $Y^2 \leq Y$.

Now $(WY)^2 = (WY)(WY) = W(YW)Y = W(WY)Y = (WW)(YY) = W^2Y^2$,

i.e., $(WY)^2 \leq (WY)$. Hence WY is transitive.

Remark 26. *If W is a transitive mPFM in M_m, then W^k is transitive for any $k \in N$.*

Proposition 27. *If $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \right\rangle\right] \in M_m$ is symmetric and transitive, then $q_{lk} \leq q_{ll}$ for $0 \leq l, k \leq m$.*

Proof. Since Q is symmetric $[q_{lk}] = [q_{kl}]$ for all $0 \leq l, k \leq m$. Also since Q is transitive $Q^2 \leq Q$,

i.e., $Q \geq Q^2$. Thus for $j \in \{1, 2, \cdots, m\}$, $q_{lk} \geq \max_j \{\min(q_{lj}, q_{jk})\}$ for all l, k,

i.e., $q_{ll} \geq \max_j \{\min(q_{lj}, q_{jl})\}$ for $l = k$ for each j

$\geq \min(q_{lk}, q_{kl})$ for $j = k$ for each l.

This implies that $q_{ll} \geq q_{lk}$ [Since $q_{lk} \geq q_{kl}$].

5. Eigenvalues and Eigenvectors of m-polar fuzzy matrices

In many areas, E. Value problems play a major role. These concepts are very helpful in mathematical modeling of real situations. For instance, the natural frequencies and normal mode shapes in free vibration of a two mass systems related problems, the axes of principal in elasticity and dynamics, the Markov chain rule in the modeling of stochastic and in queuing theory, and in the process of analytical hierarchy for decision making, etc. all give up using E. Value problems.

In this section, E. Values and E. Vectors of an mPFM using max-min operation is defined and some of its properties are studied.

Definition 28. Let $Q \in M_m$ and a scalar $\lambda = \langle \lambda_1, \lambda_2, \cdots, \lambda_m \rangle \in M_F$ is an E. Value of Q and a vector $X \neq 0$ is a row (column) E. Vector of Q if $XQ = \lambda X$ $(QX = \lambda X)$, X is called an *E. Vector with respect to the E. Value* λ.

Theorem 29. *If* $Q = [q_{lk}] = \left[\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \rangle\right]$ *is a square mPFM of order* m *, such that* $q_{1l,} = q_{2l} = \cdots = q_{l-1,l} = q_{l+1,l} = \cdots = q_{ml} = o_m$ *(say) where* $0 \le l \le m$*. Then* q_{ll} *is an E. Value with respect to the column E. Vector* $[o_m, o_m, \cdots, i_m, \cdots o_m]^T \in S^m$*, where* $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$ *be the* l *th entry.*

Proof. Here $X = [o_m, o_m, \cdots, i_m, \cdots o_m]^T = (y_{l1}) \in S^m$ (say). Then

$$QX = \begin{bmatrix} \sum_{j=1}^{m} q_{1j} y_{j1} \\ \sum_{j=1}^{m} q_{2j} y_{j1} \\ \vdots \\ \sum_{j=1}^{m} q_{mj} y_{j1} \end{bmatrix} = \begin{bmatrix} o_m \\ o_m \\ \vdots \\ q_{ll} \\ \vdots \\ o_m \end{bmatrix} = q_{ll} \begin{bmatrix} o_m \\ o_m \\ \vdots \\ i_m \\ \vdots \\ o_m \end{bmatrix}$$

[Since l th entry

$$\sum_{j=1}^{m} q_{lj} y_{j1} = q_{l1} \cdot o_m + q_{l2} \cdot o_m + \cdots + q_{ll} \cdot i_m + \cdots + q_{lm} \cdot o_m = q_{ll} \cdot o_m + q_{ll} \cdot o_m + \cdots + q_{ll} \cdot i_m + \cdots + q_{ll} \cdot o_m .]$$

Therefore, $QX = q_{ll} X$.

Hence q_{ll} *is an E. Value with respect to the column E. Vector* $[o_m, o_m, \cdots, i_m, \cdots o_m]^T \in S^m$.

Example 30. Let $Q = \begin{bmatrix} \langle 0.3, 0.4, 0.8 \rangle & \langle 0.0, 0.0, 0.0 \rangle & \langle 0.8, 0.1, 0.7 \rangle \\ \langle 0.5, 0.4, 0.6 \rangle & \langle 0.5, 0.6, 0.8 \rangle & \langle 0.6, 0.4, 0.1 \rangle \\ \langle 0.3, 0.4, 0.2 \rangle & \langle 0.0, 0.0, 0.0 \rangle & \langle 0.8, 0.7, 0.1 \rangle \end{bmatrix}$ and

$$X = \begin{bmatrix} \langle 0.0, 0.0, 0.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle \end{bmatrix}.$$

Then $$QX = \begin{bmatrix} \langle 0.3, 0.4, 0.8 \rangle & \langle 0.0, 0.0, 0.0 \rangle & \langle 0.8, 0.1, 0.7 \rangle \\ \langle 0.5, 0.4, 0.6 \rangle & \langle 0.5, 0.6, 0.8 \rangle & \langle 0.6, 0.4, 0.1 \rangle \\ \langle 0.3, 0.4, 0.2 \rangle & \langle 0.0, 0.0, 0.0 \rangle & \langle 0.8, 0.7, 0.1 \rangle \end{bmatrix} \odot \begin{bmatrix} \langle 0.0, 0.0, 0.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.0, 0.0, 0.0 \rangle \\ \langle 0.5, 0.6, 0.8 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle \end{bmatrix} = \langle 0.5, 0.6, 0.8 \rangle \begin{bmatrix} \langle 0.0, 0.0, 0.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle \end{bmatrix} = \langle 0.5, 0.6, 0.8 \rangle X .$$

Thus, $\langle 0.5, 0.6, 0.8 \rangle$ is the E. Value of Q with respect to the column E. Vector X .

From Theorem 29. and Example 30. , we have the following

Note 31. Let $q_{ll} = \langle q_{1_{ll}}, q_{2_{ll}}, \cdots, q_{m_{ll}} \rangle$, $\alpha = \langle \alpha_1, \alpha_2, \cdots, \alpha_m \rangle \in M_F$ and if $q_{ll} \le \alpha$, i.e., if $q_{1_{ll}} \le \alpha_1$, $q_{2_{ll}} \le \alpha_2$ and $q_{m_{ll}} \le \alpha_m$ then $[o_m, o_m, \cdots, i_m, \cdots o_m]^T = \alpha[o_m, o_m, \cdots, i_m, \cdots o_m]^T \in S^m$ are also E. Vectors with respect to the same E. Value q_{ll}, for any scalar $\alpha \in M_F$. So it is observed that E. Vectors with respect to the same E. Value are not unique.

Theorem 32. *If $Q = [q_{lk}] = [\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \rangle]$ is a square mPFM of order m such that $q_{l1,} = q_{l2} = \cdots = q_{l,\,l-1} = q_{l,\,l+1} = \cdots = q_{lm} = o_m$ (say) where $0 \le l \le m$. Then q_{ll} is an E. Value with respect to the row E. Vector $[o_m, o_m, \cdots, i_m, \cdots o_m] \in S_m$, where $i_m = \langle 1.0, 1.0, \cdots, 1.0 \rangle$ be the lth entry. In addition, $q_{ll} \le \alpha$ for some $\alpha \in M_F$, then $\alpha[o_m, o_m, \cdots, i_m, \cdots o_m] \in S_m$ are also E. Vectors with respect to the same E. Value q_{ll}.*

Proof. Here $X = [o_m, o_m, \cdots, i_m, \cdots o_m] = (y_{1l}) \in S_m$ (say). Then

$$XQ = \left[\sum_{j=1}^{m} y_{1j} q_{j1} \quad \sum_{j=1}^{m} y_{1j} q_{j2} \quad \cdots \quad \sum_{j=1}^{m} y_{1j} q_{jm} \right]$$

$$= [o_m, o_m, \cdots, i_m, \cdots o_m] = q_{ll}[o_m, o_m, \cdots, i_m, \cdots o_m]$$

[Since lth entry

$$\sum_{j=1}^{m} y_{1j} q_{j1} = q_{1l} \cdot o_m + q_{2l} \cdot o_m + \cdots + q_{ll} \cdot i_m + \cdots + q_{ml} \cdot o_m = q_{ll} \cdot o_m + q_{ll} \cdot o_m + \cdots + q_{ll} \cdot i_m + \cdots + q_{ll} \cdot o_m .]$$

Therefore, $XQ = q_{ll} X$.

Hence q_{ll} *is an E. Value with respect to the row E. Vector* $[o_m, o_m, \cdots, i_m, \cdots o_m] \in S_m$.

Example 33. Let $Q = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.4, 0.6, 0.7 \rangle & \langle 0.8, 0.9, 0.9 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle & \langle 0.4, 0.3, 0.8 \rangle & \langle 0.0, 0.0, 0.0 \rangle \\ \langle 0.5, 0.2, 0.4 \rangle & \langle 0.8, 0.9, 0.8 \rangle & \langle 0.2, 0.2, 0.2 \rangle \end{bmatrix}$ and

$X = [\langle 0.0, 0.0, 0.0 \rangle, \langle 1.0, 1.0, 1.0 \rangle, \langle 0.0, 0.0, 0.0 \rangle]$.

Then $XQ = [\langle 0.0, 0.0, 0.0 \rangle \ \langle 1.0, 01.0, 1.0 \rangle \ \langle 0.0, 0.0, 0.0 \rangle] \odot$

$$\begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.4, 0.6, 0.7 \rangle & \langle 0.8, 0.9, 0.9 \rangle \\ \langle 0.0, 0.0, 0.0 \rangle & \langle 0.4, 0.3, 0.8 \rangle & \langle 0.0, 0.0, 0.0 \rangle \\ \langle 0.5, 0.2, 0.4 \rangle & \langle 0.8, 0.9, 0.8 \rangle & \langle 0.2, 0.2, 0.2 \rangle \end{bmatrix}$$

$= \langle 0.4, 0.3, 0.8 \rangle \ X$.

Thus, $\langle 0.4, 0.3, 0.8 \rangle$ is the E. Value of Q with respect to the row E. Vector X.

Theorem 34. *If $Q = [q_{lk}] = [\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \rangle]$ is a square mPFM of order m such that $q_{1l,} = q_{2l} = \cdots = q_{ml} = \lambda \ge q_{lk}$ for all $0 \le l, k \le m$. Then λ is an E. Value with respect to the column
E. Vector $[i_m, i_m, \cdots, i_m, \cdots i_m]^T \in S^m$. In addition, $\lambda \le \alpha$ for some $\alpha \in M_F$, then $\alpha[i_m, i_m, \cdots, i_m, \cdots i_m]^T \in S^m$ are also E. Vectors with respect to the same E. Value λ.*

Proof. Since $q_{1l} = q_{2l} = \cdots = q_{ml} = \lambda \ge q_{lk}$ for all $0 \le l, k \le m$, we have $\sum_{k=1}^{m} q_{lk} = \lambda$. Also $[i_m, i_m, \cdots, i_m, \cdots i_m]^T \in S^m$. Then

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$$QX = \begin{bmatrix} \sum_{k=1}^{m} q_{1k} i_m \\ \sum_{k=1}^{m} q_{2k} i_m \\ \vdots \\ \sum_{k=1}^{m} q_{mk} i_m \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} q_{1k} \\ \sum_{k=1}^{m} q_{2k} \\ \vdots \\ \sum_{k=1}^{m} q_{mk} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} i_m \\ i_m \\ \vdots \\ i_m \end{bmatrix} = \lambda X$$

This proves that, λ is an E. Value with respect to the column E. Vector X .

Example 35. Let $Q = \begin{bmatrix} \langle 0.6, 0.7, 0.9 \rangle & \langle 0.3, 0.5, 0.2 \rangle & \langle 0.5, 0.2, 0.1 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle & \langle 0.1, 0.6, 0.4 \rangle & \langle 0.3, 0.4, 0.5 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle & \langle 0.3, 0.4, 0.5 \rangle & \langle 0.3, 0.5, 0.2 \rangle \end{bmatrix}$ and $X = \begin{bmatrix} \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \end{bmatrix}$

.

Then $QX = \begin{bmatrix} \langle 0.6, 0.7, 0.9 \rangle & \langle 0.3, 0.5, 0.2 \rangle & \langle 0.5, 0.2, 0.1 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle & \langle 0.1, 0.6, 0.4 \rangle & \langle 0.3, 0.4, 0.5 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle & \langle 0.3, 0.4, 0.5 \rangle & \langle 0.3, 0.5, 0.2 \rangle \end{bmatrix} \odot \begin{bmatrix} \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \end{bmatrix}$

$$= \begin{bmatrix} \langle 0.6, 0.7, 0.9 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle \\ \langle 0.6, 0.7, 0.9 \rangle \end{bmatrix} = \langle 0.6, 0.7, 0.9 \rangle \begin{bmatrix} \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \\ \langle 1.0, 1.0, 1.0 \rangle \end{bmatrix} = \langle 0.6, 0.7, 0.9 \rangle \; X .$$

Thus, $\langle 0.6, 0.7, 0.9 \rangle$ is the column E. Value of Q with respect to the E. Vector X.

Theorem 36. *If $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}}, q_{2_{lk}}, \cdots, q_{m_{lk}} \right\rangle\right]$ is a square mPFM of order m such that $q_{l1,} = q_{l2} = \cdots = q_{lm} = \lambda \geq q_{lk}$ for all $0 \leq l, k \leq m$. Then λ is an E. Value of Q with respect to the row E. Vector $[i_m, i_m, \cdots, i_m, \cdots i_m] \in S_m$. In addition, $\lambda \leq \alpha$ for some $\alpha \in M_F$, then $\alpha[i_m, i_m, \cdots, i_m, \cdots i_m] \in S_m$ are also E. Vectors with respect to the same E. Value λ.*

Proof. Since $q_{l1,} = q_{l2} = \cdots = q_{lm} = \lambda \geq q_{lk}$ for all $0 \leq l, k \leq m$, we have $\sum_{k=1}^{m} q_{kl} = \lambda$. Also $[i_m, i_m, \cdots, i_m, \cdots i_m] \in S_m$. Then

$$XQ = \begin{bmatrix} \sum_{k=1}^{m} q_{k1} i_m & \sum_{k=1}^{m} q_{k2} i_m & \cdots & \sum_{k=1}^{m} q_{km} i_m \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} q_{k1} & \sum_{k=1}^{m} q_{k2} & \cdots & \sum_{k=1}^{m} q_{km} \end{bmatrix} = [\lambda \quad \lambda \quad \cdots \quad \lambda]$$

$$= \lambda [i_m, i_m, \cdots, i_m, \cdots i_m] = \lambda X.$$

This shows that, λ is an E. Value with respect to the row E. Vector X .

Example 37. Let $Q = \begin{bmatrix} \langle 0.7, 0.9, 0.6 \rangle & \langle 0.7, 0.9, 0.6 \rangle & \langle 0.7, 0.9, 0.6 \rangle \\ \langle 0.6, 0.8, 0.5 \rangle & \langle 0.1, 0.2, 0.3 \rangle & \langle 0.2, 0.1, 0.5 \rangle \\ \langle 0.5, 0.8, 0.3 \rangle & \langle 0.5, 0.5, 0.3 \rangle & \langle 0.1, 0.5, 0.3 \rangle \end{bmatrix}$ and

$X = [\langle 1.0, 1.0, 1.0 \rangle \quad \langle 1.0, 1.0, 1.0 \rangle \quad \langle 1.0, 1.0, 1.0 \rangle]$.

Then $XQ = [\langle 1.0, 1.0, 1.0 \rangle \quad \langle 1.0, 1.0, 1.0 \rangle \quad \langle 1.0, 1.0, 1.0 \rangle] \odot$

$$\begin{bmatrix} \langle 0.7,0.9,0.6\rangle & \langle 0.7,0.9,0.6\rangle & \langle 0.7,0.9,0.6\rangle \\ \langle 0.6,0.8,0.5\rangle & \langle 0.1,0.2,0.3\rangle & \langle 0.2,0.1,0.5\rangle \\ \langle 0.5,0.8,0.3\rangle & \langle 0.5,0.5,0.3\rangle & \langle 0.1,0.5,0.3\rangle \end{bmatrix}$$

$$= \left[\langle 0.7,0.9,0.6\rangle \quad \langle 0.7,0.9,0.6\rangle \quad \langle 0.7,0.9,0.6\rangle\right]$$

$$= \langle 0.7,0.9,0.6\rangle \left[\langle 1.0,1.0,1.0\rangle \quad \langle 1.0,1.0,1.0\rangle \quad \langle 1.0,1.0,1.0\rangle\right] = \langle 0.7,0.9,0.6\rangle \; X$$

Thus, $\langle 0.7,0.9,0.6\rangle$ is the E. Value of Q with respect to the row E. Vector X .

Definition 38. (Diagonally dominant) Let $Q=[q_{lk}]=\left[\left\langle q_{1_{lk}},\, q_{2_{lk}},\cdots,q_{m_{lk}}\right\rangle\right]$ be a square mPFM of order m. Then Q is called *row diagonally dominant* if $q_{ll} \geq \sum_{k\neq l,\,k=1}^{m} q_{lk}$. Q is called *column diagonally dominant* if $q_{ll} \geq \sum_{l\neq k,\,l=1}^{m} q_{lk}$. Q is called *diagonally dominant* if it is both row and column diagonally dominant.

Theorem 39. *Let* $Q=[q_{lk}]=\left[\left\langle q_{1_{lk}},\, q_{2_{lk}},\cdots,q_{m_{lk}}\right\rangle\right]\in M_m$ *be an mPFM such that* $q_{11}=q_{22}=\cdots=q_{mm}=t$ *(say) and if* Q *is called diagonally dominant, then* t *is an E. Value with respect to the row (column) E. Vectors* $\alpha\left(i_m,\, i_m,\, i_m,\,\cdots,\, i_m\right)\in S_m$ $\left(\alpha\left[i_m,\, i_m,\, i_m,\,\cdots,\, i_m\right]^T \in S^m\right)$ *for some* $\alpha\in M_F$ *with* $t\leq\alpha$.

Proof. Since an mPFM $Q=[q_{lk}]$ is diagonally dominant, we have $\sum_{k=1}^{m} q_{lk}=q_{ll}=t$ and $\sum_{l=1}^{m} q_{lk}=q_{kk}=t$. Also $\alpha\left[i_m,\, i_m,\, i_m,\,\cdots,\, i_m\right]^T\in S^m$. Then

$$QX=\begin{bmatrix}\alpha\sum_{k=1}^{m} q_{1k}i_m \\ \alpha\sum_{k=1}^{m} q_{2k}i_m \\ \vdots \\ \alpha\sum_{k=1}^{m} q_{mk}i_m\end{bmatrix}=\begin{bmatrix}\alpha\sum_{k=1}^{m} q_{1k} \\ \alpha\sum_{k=1}^{m} q_{2k} \\ \vdots \\ \alpha\sum_{k=1}^{m} q_{mk}\end{bmatrix}=\begin{bmatrix}\alpha t\\ \alpha t\\ \vdots\\ \alpha t\end{bmatrix}=t\alpha\begin{bmatrix}i_m\\ i_m\\ \vdots\\ i_m\end{bmatrix}=tX\,.$$

Thus, t is an E. Value of an mPFM Q with respect to the column vectors X .
Similarly, we can prove the theorem for row E. Vectors.

$$XQ=\left[\alpha\sum_{k=1}^{m} q_{k1}i_m \quad \alpha\sum_{k=1}^{m} q_{k2}i_m \quad \cdots \quad \alpha\sum_{k=1}^{m} q_{km}i_m\right]$$

$$=\left[\alpha\sum_{k=1}^{m} q_{k1} \quad \alpha\sum_{k=1}^{m} q_{k2} \quad \cdots \quad \alpha\sum_{k=1}^{m} q_{km}\right]$$

$$=\left[\alpha t \quad \alpha t \quad \cdots \quad \alpha t\right]=t\alpha\left[i_m,\, i_m,\,\cdots,\, i_m\right]=tX.$$

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Example 40. Let $Q=\begin{bmatrix}\langle 0.8,0.5,\ 0.6\rangle & \langle 0.3,0.4,\ 0.5\rangle & \langle 0.7,0.2,\ 0.5\rangle\\ \langle 0.6,0.3,\ 0.4\rangle & \langle 0.8,0.5,\ 0.6\rangle & \langle 0.6,0.4,\ 0.3\rangle\\ \langle 0.1,0.4,\ 0.5\rangle & \langle 0.6,0.1,\ 0.1\rangle & \langle 0.8,0.5,\ 0.6\rangle\end{bmatrix}$ and

$X=\left[\langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle\right]$.

Then $XQ=\left[\langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle\right]\odot$

$$\begin{bmatrix}\langle 0.8,0.5,\ 0.6\rangle & \langle 0.3,0.4,\ 0.5\rangle & \langle 0.7,0.2,\ 0.5\rangle\\ \langle 0.6,0.3,\ 0.4\rangle & \langle 0.8,0.5,\ 0.6\rangle & \langle 0.6,0.4,\ 0.3\rangle\\ \langle 0.1,0.4,\ 0.5\rangle & \langle 0.6,0.1,\ 0.1\rangle & \langle 0.8,0.5,\ 0.6\rangle\end{bmatrix}$$

$$=\left[\langle 0.8,0.5,\ 0.6\rangle \quad \langle 0.8,0.5,\ 0.6\rangle \quad \langle 0.8,0.5,\ 0.6\rangle\right]$$

$$=\langle 0.8,0.5,\ 0.6\rangle\left[\langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle \quad \langle 0.9,0.8,0.7\rangle\right]=\langle 0.8,0.5,\ 0.6\rangle\ X$$

Thus, $\langle 0.8,0.5,\ 0.6\rangle$ is the E. Value of Q with respect to the row E. Vector X .

Example 41. let $Q=\begin{bmatrix}\langle 0.8,0.5,\ 0.6\rangle & \langle 0.3,0.4,\ 0.5\rangle & \langle 0.7,0.2,\ 0.5\rangle\\ \langle 0.6,0.3,\ 0.4\rangle & \langle 0.8,0.5,\ 0.6\rangle & \langle 0.6,0.4,\ 0.3\rangle\\ \langle 0.1,0.4,\ 0.5\rangle & \langle 0.6,0.1,\ 0.1\rangle & \langle 0.8,0.5,\ 0.6\rangle\end{bmatrix}$ and

$X=\begin{bmatrix}\langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\end{bmatrix}$.

Then $QX=\begin{bmatrix}\langle 0.8,0.5,\ 0.6\rangle & \langle 0.3,0.4,\ 0.5\rangle & \langle 0.7,0.2,\ 0.5\rangle\\ \langle 0.6,0.3,\ 0.4\rangle & \langle 0.8,0.5,\ 0.6\rangle & \langle 0.6,0.4,\ 0.3\rangle\\ \langle 0.1,0.4,\ 0.5\rangle & \langle 0.6,0.1,\ 0.1\rangle & \langle 0.8,0.5,\ 0.6\rangle\end{bmatrix}\odot\begin{bmatrix}\langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\end{bmatrix}$

$$=\begin{bmatrix}\langle 0.8,0.5,\ 0.6\rangle\\ \langle 0.8,0.5,\ 0.6\rangle\\ \langle 0.8,0.5,\ 0.6\rangle\end{bmatrix}=\langle 0.8,0.5,\ 0.6\rangle\begin{bmatrix}\langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\\ \langle 0.9,0.8,0.7\rangle\end{bmatrix}$$

Thus, $\langle 0.8,0.5,\ 0.6\rangle$ is the E. Value of Q with respect to the column E. Vector X .

Theorem 42. *Let* $Q=[q_{lk}]=\left[\left\langle q_{1_{lk}},\ q_{2_{lk}},\cdots,q_{m_{lk}}\right\rangle\right]\in M_m$ *be an mPFM then* $\lambda=(\lambda_1,\ \lambda_2,\cdots,\lambda_m)\in M_F$ *be an E. Value with respect to the column E. Vectors* $\left(\alpha\left[i_m,\ i_m,\ i_m,\ \cdots,\ i_m\right]^T\right)\in S^m$ *if* $\max\left\{q_{1_{s1}},\ q_{1_{s2}},\ \cdots,q_{1_{sm}}\right\}=\lambda_1$, $\max\left\{q_{2_{s1}},\ q_{2_{s2}},\ \cdots,q_{2_{sm}}\right\}=\lambda_2$ *and* $\max\left\{q_{m_{s1}},\ q_{m_{s2}},\ \cdots,q_{m_{sm}}\right\}=\lambda_m$ *for every* $s\in\{1,\ 2,\ \cdots,\ m\}$ *and for some* $\alpha\in M_F$ *with* $\lambda\le\alpha$.

Proof. Since $\max\left\{q_{1_{s1}},\ q_{1_{s2}},\ \cdots,q_{1_{sm}}\right\}=\lambda_1$, $\max\left\{q_{2_{s1}},\ q_{2_{s2}},\ \cdots,q_{2_{sm}}\right\}=\lambda_2$ and $\max\left\{q_{m_{s1}},\ q_{m_{s2}},\ \cdots,q_{m_{sm}}\right\}=\lambda_m$ for every $s\in\{1,\ 2,\ \cdots,\ m\}$, we have

$$\sum_{k=1}^{m}q_{sk}=\left(\sum_{k=1}^{m}q_{1_{sk}},\ \sum_{k=1}^{m}q_{2_{sk}},\cdots,\sum_{k=1}^{m}q_{m_{sk}}\right)=(\lambda_1,\ \lambda_2,\cdots,\lambda_m)=\lambda \text{ for every } s\in\{1,\ 2,\ \cdots,\ m\}.$$

Also, $\left(\alpha\left[i_m,\, i_m,\, i_m,\, \cdots,\, i_m\right]^T\right) \in S^m$. Then

$$QX = \begin{bmatrix} \alpha\sum_{k=1}^{m} q_{1k} i_m \\ \alpha\sum_{k=1}^{m} q_{2k} i_m \\ \vdots \\ \alpha\sum_{k=1}^{m} q_{mk} i_m \end{bmatrix} = \begin{bmatrix} \alpha\sum_{k=1}^{m} q_{1k} \\ \alpha\sum_{k=1}^{m} q_{2k} \\ \vdots \\ \alpha\sum_{k=1}^{m} q_{mk} \end{bmatrix} = \begin{bmatrix} \alpha\lambda \\ \alpha\lambda \\ \vdots \\ \alpha\lambda \end{bmatrix} = \lambda\alpha \begin{bmatrix} i_m \\ i_m \\ \vdots \\ i_m \end{bmatrix} = \lambda X$$

Thus, λ is an E. Value of an mPFM Q with respect to the column Vectors X.

Example 43. Let $Q = \begin{bmatrix} \langle 0.7,0.1,0.4\rangle & \langle 0.5,0.9,0.3\rangle & \langle 0.6,0.5,0.4\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 0.4,0.9,0.2\rangle & \langle 0.7,0.3,0.4\rangle \\ \langle 0.5,0.3,0.4\rangle & \langle 0.7,0.2,0.1\rangle & \langle 0.1,0.9,0.1\rangle \end{bmatrix}$ and $X = \begin{bmatrix} \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \end{bmatrix}$

.

Then $QX = \begin{bmatrix} \langle 0.7,0.1,0.4\rangle & \langle 0.5,0.9,0.3\rangle & \langle 0.6,0.5,0.4\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 0.4,0.9,0.2\rangle & \langle 0.7,0.3,0.4\rangle \\ \langle 0.5,0.3,0.4\rangle & \langle 0.7,0.2,0.1\rangle & \langle 0.1,0.9,0.1\rangle \end{bmatrix} \odot \begin{bmatrix} \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \end{bmatrix}$

$$= \begin{bmatrix} \langle 0.7,\, 0.9,0.4\rangle \\ \langle 0.7,\, 0.9,0.4\rangle \\ \langle 0.7,\, 0.9,0.4\rangle \end{bmatrix} = \langle 0.7,\, 0.9,0.4\rangle \begin{bmatrix} \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \\ \langle 1.0,\, 1.0,1.0\rangle \end{bmatrix} = \langle 0.7,\, 0.9,0.4\rangle\ X.$$

Thus, $\langle 0.7,\, 0.9,0.4\rangle$ is the column E. Value of Q with respect to the E. Vector X.

Theorem 44. *Let $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}},\, q_{2_{lk}}, \cdots, q_{m_{lk}}\right\rangle\right] \in M_m$ be an mPFM then $\lambda = (\lambda_1,\, \lambda_2, \cdots, \lambda_m) \in M_F$ be an E. Value with respect to the row E. Vectors $\left(\alpha\left[i_m,\, i_m,\, i_m,\, \cdots,\, i_m\right]\right) \in S_m$ if $\max\left\{q_{1_{1s}},\, q_{1_{2s}},\, \cdots, q_{1_{ms}}\right\} = \lambda_1$, $\max\left\{q_{2_{1s}},\, q_{2_{2s}},\, \cdots, q_{2_{ms}}\right\} = \lambda_2$ and $\max\left\{q_{m_{1s}},\, q_{m_{2s}},\, \cdots, q_{m_{ms}}\right\} = \lambda_m$ for every $s \in \{1,\, 2,\, \cdots,\, m\}$ and for some $\alpha \in M_F$ with $\lambda \le \alpha$.*

Proof. Since $\max\left\{q_{1_{1s}},\, q_{1_{2s}},\, \cdots, q_{1_{ms}}\right\} = \lambda_1$, $\max\left\{q_{2_{1s}},\, q_{2_{2s}},\, \cdots, q_{2_{ms}}\right\} = \lambda_2$ and $\max\left\{q_{m_{1s}},\, q_{m_{2s}},\, \cdots, q_{m_{ms}}\right\} = \lambda_m$ for every $s \in \{1,\, 2,\, \cdots,\, m\}$, we have

$$\sum_{k=1}^{m} q_{ks} = \left(\sum_{k=1}^{m} q_{1_{ks}},\, \sum_{k=1}^{m} q_{2_{ks}}, \cdots, \sum_{k=1}^{m} q_{m_{ks}}\right) = (\lambda_1,\, \lambda_2, \cdots, \lambda_m) = \lambda \text{ for every } s \in \{1,\, 2,\, \cdots,\, m\}.$$

Also, $\left(\alpha\left[i_m,\, i_m,\, i_m,\, \cdots,\, i_m\right]\right) \in S_m$. Then

$$XQ = \left[\alpha\sum_{k=1}^{m} q_{k1} i_m \quad \alpha\sum_{k=1}^{m} q_{k2} i_m \quad \cdots \quad \alpha\sum_{k=1}^{m} q_{km} i_m\right] = \left[\alpha\lambda \quad \alpha\lambda \quad \cdots \quad \alpha\lambda\right]$$

$$= \lambda\alpha\left[i_m,\, i_m,\, i_m,\, \cdots,\, i_m\right] = \lambda X.$$

Thus, λ is an E. Value of an mPFM Q with respect to the row Vectors X.

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Example 45. Let $Q = \begin{bmatrix} \langle 0.7, 0.1, 0.4 \rangle & \langle 0.5, 0.2, 0.3 \rangle & \langle 0.6, 0.5, 0.3 \rangle \\ \langle 0.6, 0.2, 0.3 \rangle & \langle 0.4, 0.9, 0.2 \rangle & \langle 0.7, 0.3, 0.2 \rangle \\ \langle 0.5, 0.4, 0.1 \rangle & \langle 0.7, 0.2, 0.1 \rangle & \langle 0.1, 0.2, 0.4 \rangle \end{bmatrix}$ and

$X = \left[\langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \right]$.

Then $XQ = \left[\langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \right]$

$$\odot \begin{bmatrix} \langle 0.6, 0.1, 0.4 \rangle & \langle 0.5, 0.8, 0.3 \rangle & \langle 0.6, 0.5, 0.3 \rangle \\ \langle 0.5, 0.8, 0.3 \rangle & \langle 0.6, 0.7, 0.2 \rangle & \langle 0.2, 0.3, 0.9 \rangle \\ \langle 0.5, 0.4, 0.9 \rangle & \langle 0.4, 0.2, 0.9 \rangle & \langle 0.1, 0.8, 0.4 \rangle \end{bmatrix}$$

$= \left[\langle 0.6, 0.8, 0.9 \rangle \right] = \langle 0.6, 0.8, 0.9 \rangle \left[\langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \quad \langle 1.0,\ 1.0, 1.0 \rangle \right]$

$= \langle 0.6, 0.8, 0.9 \rangle \ X$

Thus, $\langle 0.6, 0.8, 0.9 \rangle$ is the E. Value of Q with respect to the row E. Vector X.

Corollary 46. *Let $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}},\ q_{2_{lk}}, \cdots, q_{m_{lk}} \right\rangle \right] \in M_m$ be an mPFM. If $\sum_{s=1}^{m} q_{1s} = \sum_{s=1}^{m} q_{2s} = \cdots = \sum_{s=1}^{m} q_{ms} = t$ (say). Then, t is an E. Value of Q with respect to the column E. Vectors $\left(\alpha \left[i_m,\ i_m,\ i_m,\ \cdots,\ i_m \right]^T \right) \in S^m$ for some $\alpha \in M_F$ with $t \le \alpha$.*

Corollary 47. *Let $Q = [q_{lk}] = \left[\left\langle q_{1_{lk}},\ q_{2_{lk}}, \cdots, q_{m_{lk}} \right\rangle \right] \in M_m$ be an mPFM. If $\sum_{s=1}^{m} q_{s1} = \sum_{s=1}^{m} q_{s2} = \cdots = \sum_{s=1}^{m} q_{sm} = t$ (say). Then, t is an E. Value of Q with respect to the row E. Vectors $\alpha \left[i_m,\ i_m,\ i_m, \cdots,\ i_m \right] \in S_m$ for some $\alpha \in M_F$ with $t \le \alpha$.*

Theorem 48. *Let $Q \in M_m$, then Q has a zero column if and only if $o_m \in \sigma(Q)$ (set of all E. Values of Q).*

Proof. Condition is necessary: Let lth column of Q is zero, we take $\left[o_m,\ o_m,\ \cdots, i_m,\ \cdots\ o_m \right]^T \in S^m$, where i_m is the lth entry, then X is a non-zero vector satisfying the equation $QX = o_m X$. Hence, X is a column E. Vector with respect to the E. Value o_m.

Condition is sufficient: Let $X = \left[p_1,\ p_2,\ \cdots,\ p_m \right]^T \in S^m$ be a column E. Vector with respect to the E. Value o_m, then $QX = o_m X$. We assume that $p_l \ne o_m$ for $l \in \{1,\ 2,\ \cdots,\ m\}$. Then $QX = o_m X$ implies that $\sum_{s=1}^{m} q_{ks} p_s = o_m$ for each $k \in \{1,\ 2,\ \cdots,\ m\}$. This implies that $q_{ks} p = o_m$ for each s and k. Since $p_l \ne o_m$, $q_{lk} = o_m$ for each k, then lth entry of Q is zero.

Definition 49. *Let* $\sigma(Q)$ be the set of all E. Values of Q. Then $\delta(Q) = \sup\{\lambda | \lambda \in \sigma(Q)\}$ is called the *spectral radius* of Q.

Theorem 50. *Let $Q \in M_m$. Then $\delta(Q)$ is either o_m or i_m.*

Proof. If $\sigma(Q) = \{o_m\}$, then $\delta(Q) = o_m$, otherwise, if there exist $\lambda \in \sigma(Q)$ ($\lambda \ne o_m$) then there is a non-zero E. vector $X \in S^m$ (set of column vectors of order m) such that $QX = \lambda X$. Also we

know that for any γ with $\lambda \le \gamma \le i_m$, $\gamma \cdot \lambda = \lambda$ and $\lambda \cdot \lambda = \lambda$. Therefore,
$\lambda X = (\gamma \cdot \lambda) X = \gamma(\lambda X) \Rightarrow Q(\lambda X) = \lambda(QX) = \lambda(\lambda X) = (\lambda \cdot \lambda) X = \lambda X = \gamma(\lambda X)$.
Hence, $\gamma \in \sigma(Q)$. Since γ is arbitrary, $i_m \in \sigma(Q)$. Therefore $\delta(Q) = i_m$.

Theorem 51. *For any* $P, Q \in M_m$ *if* $P \le Q$ *then* $\delta(P) \le \delta(Q)$.

Proof. From Theorem 50, $\delta(P)$ is either o_m or i_m. If $\delta(P) = o_m$, then $\delta(P) \le \delta(Q)$ holds trivially. If $\delta(P) = i_m$, we have to prove that $\delta(Q) = i_m$. Since $\delta(P) = i_m$, then by definition $i_m \in \sigma(P)$ and $PX = i_m X = X$ for some non-zero column vector X.
We consider $e = [i_m, i_m, \cdots, i_m]^T \in S^m$. Then $X \le e$.
Also, $P^m X = P^{m-1} PX = P^{m-1} X = P^{m-2} PX = P^{m-2} X = \cdots = P^2 X = PX = X$,
i.e., $X = P^m X \le P^m e \le Q^m e$. [Since $X \le e$ and $P \le Q$.]
Since X is non-zero $Q^m e$ is non-zero. Now, if $R = Q^m e$, then $QR = Q^{m+1} e = Q^m e = R = i_m R$.
Hence $i_m \in \sigma(Q)$. Thus, $\delta(Q) = i_m$. Therefore $\delta(P) \le \delta(Q)$.

Conclusions

Similarity relations between mPFMs and properties of E. Values and E. Vectors of mPFMs are studied. Many works are accessible to compute the E. Values and E. Vectors of a fuzzy matrix. Now, we investigated the properties of E. Values and E. Vectors of mPFMs first time in this paper, and explained with proper examples. It is observed that E. Vectors with respect to an E. Value are not unique for an mPFM. Though the proposed theorems are not established for general cases. Further, the work can be extended to study the nature of the quadratic forms of mPFMs.

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