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A Compact Review – Mathematical Modeling of Rail Track

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Abstract:

The structure of the Railway Track consists of (i) Super-Structure - the Rail, Fastening System and Sleepers; (ii) Sub-Structure - Non-Cohesive Granular Materials viz Stone Ballast. The Railway Track is a Continuous Beam Structure that is supported on the Elastic Foundation having Sleepers, Rail Pad, Liners and Elastic Clips tied to the Rail.

This review paper describes the efforts done by researchers to identify the Mathematical Behavior of the Rail or the Rail Track. It gives an overview of the Static and Dynamic Stiffness of the Track and provides the global picture on the Analytical and Numerical Track Modeling. The ***Zimmermann*** *Model with Beam on Elastic Foundation (BOEF) is most widely used in the one parameter, linear models. The Pasternak foundation is popular among the complex mathematical formulations that uses moving mass or moving loads. The dynamic behavior of tracks with multi-parameter models, non-linear behavior of the foundation is employed and studied by various researchers, one can choose it based on the different boundary conditions.*

Keywords: *Rail Track, Track Modulus, Beam on Continuous Elastic Foundation – EOEF, Analytical Modeling of Track, Numerical Modeling of Track, Winkler Foundation*

1. Introduction: Why the Modeling of Track is important?

The structure of the Railway Track consists of (i) Super-Structure - the Rail, Fastening System and Sleepers; (ii) Sub-Structure - non-cohesive Granular Materials viz stone Ballast.

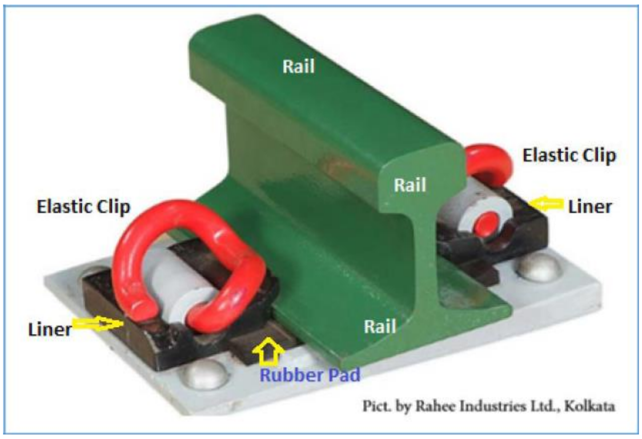

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Fig.1 A schematic of Rail Track and Rail Fastening System
[Source: http://www.pandrolrahee.com/]

Till date, there are thousands of research papers published on Rail. Every year, there is continuous addition of papers. These continuous efforts of research is useful to understand and improve the rail (combining the rail track, super and sub structures) performances, track geometries, stresses in rails, Modal Testing of rail, dynamics of rail (in terms of vertical / lateral vibration, natural frequencies, stiffness, and noise), rail-track interaction, contact mechanism, rail joints, welds, life of rail and its components, maintenance of rail, track deterioration. In return, it improves the comfort of passengers. There are few dedicated codes (software) like GEOTRACK, TRACK, DYNATACK, DARTS, DIFF, DIFF3D, NUCARS, ADAM/RAIL, MEDYNA, GENSYS, SIMPACK, VAMPIRE etc…

Railway track stiffness (vertical track load divided by track deflection) is a basic parameter of track design which influences the bearing capacity, the dynamic behavior of passing vehicles and, track geometry quality and the life of track components. The Dynamic track modelling is needed to allow more accurate prediction of track degradation and associated Railway Track maintenance and structural capacity of Railway Track and its components.

Modal Testing Experimental Modal Analysis (EMA) or modal testing is a non-destructive testing strategy based on the response of structures to vibration. Since the 1940s, modal testing has been widely used to help understand the dynamic behavior of structures. The original modal testing technique was based on the simple sine dwell method. After some years, innovations based on Fast Fourier Transform (FFT) have been developed and are currently used (Brown, 1982; Allemang and Brown, 1986; Mitchell, 1986; Allemang, 1993; Ewin, 1995; He and Fu, 2001).

2. An early brake through on the Modeling of Track

Following is the Mathematical Modelling of Train – Track Dynamic interaction are discussed in Railway Track Dynamics - a survey by Tore Dahlberg [1], as follows:

A. Beam (Rail) on Continuous Elastic Foundation
B. Vehicle – Bridge Interaction (Moving Mass on Simply Supported Beam)
C. Beam (Rail) on Discrete Supports
D. Beam (Rail) on Discrete Supports including Ballast Model
E. Beam (Rail) on Sleepers Embedded in Continuum, including 3-D FEM Models

The dynamic behavior of beams on elastic foundations have been investigated by many researchers in engineering and especially in Railway Engineering. These studies mostly considered the Winkler elastic foundation model that consists of infinite closely-spaced linear springs [2 – 4] which is also termed as one-parameter model.

In the simplest track model described by Esveld [7], **a beam (that is a model of the rail) rests on a continuous elastic foundation**. The foundation is modelled by an evenly distributed linear spring stiffness was introduced by Winkler (1867) and is still in use for easy and quick track deflection calculations. The distributed force supporting the beam

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then is proportional to the beam deflection. The only track parameters needed for this model is the **Beam Bending Stiffness EI (N/m^2)** and the **Foundation Stiffness** (the **Bed Modulus**) **k** (**N/m^2**), i.e**. N/m per meter of rail**).

Zimmermann (1888) presented a solution for the Winkler model by assuming the rail as One – Dimensional Analysis of a railway structure supported by elastic springs at discrete points of a beam laid on a continuous support (soil subgrade or foundation). He considered the single value of **C (N/mm^3)** or modulus sub-grade reaction or ballast module is used. But the components of rail-pad, ballast, sub-ballast mat and sub soil have different C values in the ballasted track systems. Hence, the material properties of those components are combined into single **Ctot** value by using the correlation of Springs in Series.

In 1970, Meacham et. al. (1970) introduced a track model, assuming the track components to be replaced by a series of springs, with individual spring constants. Lundgren and Martin (1970), and Thompson and Tayabji (1976) considered discrete support for the rail in the static model of the track. Ahlbeck et. al (1978) simulated linear railroad track structures [based on linear theory] using one or two degrees of freedom with lumped-parameter model.

The above developments have improved the Winkler model by taking into consideration the discrete properties of the support system of the rail. However, Kerr (1976) proved that the results obtained from the discrete support model are not significantly different from those obtained from the Winkler model.

Timoshenko (1926) was one of the first to model the dynamic behavior of a railway track. In that model, the rail was considered as an infinite uniform Euler beam, laid on a continuous damped elastic Winkler foundation. Esveld [7] and Kaewunruen S., Remennikov A. [9] identified the dynamic parameters of Railway Tracks, such as natural frequency, damping constant, and corresponding vibration mode shape, are of substantial importance in the procedures needed for analysis and design of Railway Tracks. The better findings form Esveld [7], Grassie [8], Kaewunruen S. [9] for the static and dynamic responses of track component have researched to improve the capacity of existing railway Ballasted tracks and the economical analysis and design.

Grassie (1982) then found in some experiments that there are only two dominant resonances in the frequency range of interest. The first in-phase mode at about 100 Hz corresponds to the sleeper and rail moving together on the ballast while the second out-of-phase mode at a frequency somewhere between 300 – 500 Hz depending on the rail pad parameters, corresponds to the opposite vibration of sleepers on ballast and rails on the rail pad.

3. The complexity of the Modeling of Track

In the case of moving mass, studies are limited to single [13] or multiple span [15-16] beams with different boundary conditions and without any elastic supports. A very few studies considered one parameter foundation model for prediction of beam responses subjected to a moving mass [13-14]. However, this one parameter models do not accurately represent the continuous characteristics of practical foundations since it assumes no

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interaction between the lateral springs. Moreover, it also results in overlooking the influence of the soil on either side of the beam [15].

The dynamic behavior of tracks with multi-parameter models, non-linear behavior of the foundation is employed and studied by various researchers. The two-parameter models, also known as Pasternak models, have been proposed for the analysis of the dynamic behavior of beams under moving loads [21-23]. These are mathematically equivalent models but differ in foundation parameters only. However, dynamic response of the beam supported on a two-parameter foundation model under a moving mass is not investigated so far.

Moreover, the effects of shear modulus and foundation stiffness on deflection and bending moment responses of the beam supported by Pasternak foundation have also never been investigated in the presence of a moving mass.

In order to capture the distributed stresses accurately, a three-parameter model has been developed for cohesive and non-cohesive soil foundations [24-26]. This model offers the continuity in the vertical displacements at the boundaries between the loaded and the unloaded surfaces of the soil [27]. In the analysis of vibration of beams under the moving loads and masses, the beam has been modeled as either a Timoshenko beam or a Euler-Bernoulli. A Timoshenko beam model considers the shear deformation and rotational inertia of the beam.

Chen and Huang [26] have graphically shown that an Euler-Bernoulli beam can accurately predict the response of the beam for foundation stiffness up to 108 N/m2, since the depth and rotary inertia of the track can be considered small compared to the translational inertia

[28]. The analytical solution of the vibration of infinite beams under the moving load has received considerable attention by researchers. In case of two-parameter model, studies are scarce due to the model complexity and difficulties in estimating parameter values.

In recent years, there is a growing interest on the vibration of the beam under moving load in railway industry in view of the use of beam type structure as a simplified physical model for railway track and pavements. Apart from the one, two or three parameter foundation models, viscoelastic and poroelastic half space models of the foundation are also common in the dynamic analysis of a beam due to a moving oscillating load or moving point load. These half space models can be single layer, or multiple layers, Responses of the beams in terms of displacements, bending moments, accelerations, and shear force have been analyzed in these studies. Studies with multilayer half space show that the response calculated for the multi-layered case exhibits higher frequencies and larger amplitudes than the response obtained for a uniform half-space.

Fryba [2] presented a detailed solution of the moving load problem where the beam was modeled as infinitely long Euler-Bernoulli beam resting on Winkler foundation. A vast majority of the studies dealing with the moving load problems utilized the Fourier transformation method to solve the governing differential equations arising from either Euler-Bernoulli or Timoshenko theory. The responses of the infinite beam under moving load supported on either Winkler or Pasternak foundation were studied by means of Fourier transforms and using Green's function.

In order to consider the effect of non-linearity in beam analysis, Finite Element Analysis (FEA) of an infinite beam has been carried out in [32-34]. In these studies, FEA has been

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adopted to perform the analysis of nonlinear dynamic structure under moving loads where the load varies with both time and space. In moving force problem, the magnitude of the moving force has been assumed to be constant by neglecting the inertia forces of a moving mass. However, in the case of moving mass, the interaction force consists of inertia of the mass, centrifugal force, etc. Hence, the velocity of the moving mass, structural flexibility, and the ratio of the moving mass to that of the structure play important roles on the overall interaction process.

The Analytical and Numerical Modeling of Track Structure is presented in the tabular format. It describes the essential track component for the track modeling. The Track Structure considered as Single or Double Beam, presented by Euler or Timoshenko formulation, Continuously or Discretely Supported with the Suspended / Distributed Masses, either included or neglected.

Supported

Overview of Track Structure Models

Track Structure Model	**Track Structure Model**	**Rails**	**Sleepers**	**Blocks**	**Base Plates**	**Slabs**
			Mass is neglected			
			Distributed Mass			
Analytical Models	Single Beam (Euler, Timoshenko) Continously Supported	□				□
	Single Beam (Euler, Timoshenko) Continously Supported on Springs with suspended mass	□	□	□	□	
	Double Beam (Euler, Timoshenko) Continously Supported	□	Mass is neglected			□
Numerical Models	Single Beam (Timoshenko) Discretely Supported on Springs with suspended mass	□				
	Double Beam (Timoshenko) Upper Discretely Supported and Lower Continuously Supported	□				
	Double Beam (Timoshenko) Upper Continuously Supported and Lower Continuously	□				

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and Components

Rail	**Pads**	**Embede dRails**	**Plate Pads**	**Block**	**s /**	**erialS**	**oil/ d**	**ngStructu res**
Distribut ed Ma ss		□		Distributed Mass				Rigid
				Distributed Mass				Rigid
		□					□	□
	□		□		□	□	□	Rigid
			□		□	□	□	□
		□					□	□

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4. Summary and Conclusion

For the Linear Track Models (Single Spring or Equivalent Spring when Multiple Springs connected in Series), One Parameter Model, the Zimmermann (1888) model assuming the rail as One – Dimensional Analysis of a railway structure supported by elastic springs at discrete points of a beam laid on a continuous support (soil subgrade or foundation) is the most widely used.
Lundgren and Martin (1970), and Thompson and Tayabji (1976) considered discrete support for the rail in the static model of the track. Ahlbeck et. al (1978) simulated linear railroad track structures [based on linear theory] using one or two degrees of freedom with lumped-parameter model.

The Pasternak foundation is popular among the complex mathematical formulations that uses moving mass or moving loads. The dynamic behavior of tracks with multi-parameter models, non-linear behavior of the foundation is employed and studied by various researchers, one can choose it based on the different boundary conditions.

This paper is a small step to provide compact review of the efforts done by researchers to identify the Mathematical Behavior of the Rail or the Rail Track is presented in this paper. It covers both Static and Dynamic Stiffness of the Track and provides the global picture on the Analytical and Numerical Track Modeling. To model the track, one should refer to the references, for the assumptions, mathematical derivations.

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