International Journal of Disaster Recovery and Business Continuity
Vol. 12, No. 1, (2021), pp. 1541 – 1548

COMMON FIXED-POINT THEOREMS IN CONE PENTAGONAL METRIC SPACES

Shishir Jain and Pooja Chaubey
Department of Mathematics
Shri Vaishnav Vidyapeeth Vishwavidyalaya Gram Baroli Sanwer Road, Indore (M.P.) 453331, India

Abstract- *In the present paper, we prove common fixed-point theorems in cone pentagonal metric space involving two self-maps under T- Kannan and T - Reich contractions. Our results extend some previously obtained results in the settings of cone pentagonal metric spaces.*
***Keywords**: Cone pentagonal metric space; common fixed-point theorems; T-Kannan contraction; T-Reich contraction.*
***MSC 2010**: 47H10; 54H25*

1 INTRODUCTION

In 2007, Huang and Zhang [4] introduced cone metric space using ordered Banach space and proved fixed point theorems satisfying some contractive conditions in cone metric space by assuming the normality of cone. Subsequently, many researchers; see [[6], [8], [9]] studied and established fixed point and common fixed point theorems in different contractive conditions. Azam et. al. [2] in 2009 defined cone rectangular metric space and prove BCP in a complete normal cone rectangular metric space. In 2009, Jleli and Sam et [7] proved Kannan's fixed-point theorem in cone rectangular metric space assuming the condition of normality of cone. In this sequel, Garg et.al. [3] defined cone pentagonal metric spaces and established the proof of famous contraction principle named as Banach in cone pentagonal metric space. Motivated by the work of Jleli and Sam et [7], A. Auwalu [1] proved Kannan fixed point theorem in cone pentagonal metric space. Theaim of the present work is, to extend and generalize the results of A. Auwalu [1], Jleli and Sam et [7] and Kannan [5].

2 PRELIMINARIES

Definition 1. [4] Let E be a real Banach space and $P \subset E$ is a coneif and only if:
i. $P \neq \theta$, closedand $P \neq \{0\}$.
ii. $c, d \in R, c, d \geq 0$ and $y, z \in P$ implies that $cy + dz \in P$.
iii. $y \in P$ and $-y \in P$ implies that $y = 0$.

For a given cone P which is a subset of E, we defined a partialordering $\leq$ with respect to P by $y \leq z \Longleftrightarrow z - y \in P$. We couldwrite $y < z$ which implies $y \leq z$ but $y \neq z$ while $y \ll z$ will imply$z - y \in int(P)$ where the $int(P)$ denotes the interior of P. If in a cone$P$ there exists a number $K > 0$ such that for all $y, z \in E$

$$0 \leq y \leq z \Rightarrow \|y\| \leq K \|z\|$$

then P is called normal cone and normal constant of P is the leastpositive number K which satisfy the above condition.

ISSN: 2005-4289 IJDRC

In our paper we consistently suppose that E is a real Banach space, P be a solid cone and $\leq$ be a partial order with respect to P.

Definition 2. [4] Let X be a non-void set and let $d : X \times X \rightarrow E$ bea mapping such that for all $r, s, t \in X$, fascinate the below settings:

(i) $0 \leq d(r,s)$ and $d(r,s) = 0$ if and only if $r = s$;
(ii) $d(r,s) = d(s,r)$;
(iii) $d(r,s) \leq d(r,t) + d(t,s)$.

Then, the pair (X, d) is said to be cone metric space over Banach space E.

Definition 3. [3] Let X be a non-void set and let $\eta : X \times X \rightarrow E$ bea mapping such that for all $r, s \in X$, fascinate the below settings:

(i) $0 \leq \eta(r,s)$ and $\eta(r,s) = 0$ if and only if $r = s$;
(ii) $\eta(r,s) = \eta(s,r)$;
(iii) $\eta(r,s) \leq \eta(r,t) + \eta(t,w) + \eta(w,u) + \eta(u,s)$ for all $r, s, t, w, u \in X$ and for all distinct points $t, w, u \in X - \{r, s\}$ this is knownas pentagonal property.

Then, the pair (X, η) is said to be cone pentagonal metric spaceover Banach space E.

3 CRITERIA OF CONVERGENCE [3]

Definition 4. Let (X, η) be a cone pentagonal metric space and $\{u_m\} \in X$ be a sequence is said to be convergent and converge to u if for every $c \in int(P)$ there exists a positive integer $m_0$with $\eta(u_m, u) \ll c \; \forall m > m_0$. In this case u is said to be the limit of u_m, denoted by $\lim_{m \to \infty} u_m = u$.

Definition 5. Let (X, η) be a cone pentagonal metric space and $\{u_m\} \in X$ be a sequence is said to be Cauchy if $\eta(u_m, u_n) \ll c, \forall m, n > m_0$and (X, η) is called complete cone pentagonal metric space if everyCauchy sequence $\{u_m\}$ converges to $u \in X$.

Lemma 1. [3] Let (X, η) be a cone pentagonal metric space and P be anormal cone with normal constant K. Let $\{u_m\} \in X$ be a sequence issaid to be convergent and converge to u if and only if $\|\eta(u_m, u)\| \rightarrow 0$ as $m \rightarrow \infty$.

Lemma 2. [3] Let (X, η) be a cone pentagonal metric space and P bea normal cone with normal constant K. Let $\{u_m\} \in X$ be a sequenceis said to be Cauchy sequence if and only if $\|\eta(u_m, u_{m+n})\| \rightarrow 0$ as$m \rightarrow \infty$.

Example 1. Let $X = \{5, 6, 7, 8, 9\}, E = R^2$and $P = \{(u, v) : u, v \geq 0\}$is normal cone in E. Consider a map $\eta : X \times X \rightarrow E$ assigned by

$$\eta(5,7) = \eta(7,5) = \eta(7,8) = \eta(8,7) = \eta(6,7) = \eta(7,6) = \eta(6,8) = \eta(8,6) = \eta(5,8) = \eta(8,5) = (1,2)$$

$$\eta(5,6) = \eta(6,5) = (4,8)$$

$$\eta(5,9) = \eta(9,5) = \eta(6,9) = \eta(9,6) = \eta(7,9) = \eta(9,7) = \eta(8,9) = \eta(9,8) = (3,6)$$

$$\eta(u,v) = 0 \; if \; u = v.$$

ISSN: 2005-4289 IJDRC

Clearly, then (X,η) is a complete normal cone pentagonal metricspace but not a cone rectangular metric space.

4 MAIN RESULTS

Motivated from Morales and Rojas [9], we now define T – Kannan contraction and T -Reich contraction in cone pentagonal metric spacesas follows:

Definition 6. Let (X,η) be a cone pentagonal metric space and let$T, S : X \rightarrow X$ be the two self maps. Then S is called T– Kannan contraction, if there exists $\alpha \in [0,\frac{1}{2})$ such that for all $u,v \in X$,

$$\eta(TSu,TSv) \leq \alpha[\eta(Tu,TSu) + \eta(Tv,TSv)]$$

Definition 7. Let (X,η) be a cone pentagonal metric space and let$T, S : X \rightarrow X$ be the two self maps. Then S is called T– Reich contraction, if there exists $\lambda,\beta,\gamma \geq 0$ with $\lambda+\beta+\gamma < 1$ such that forall $u,v \in X$,

$$\eta(TSu,TSv) \leq \lambda\eta(Tu,TSu) + \beta\eta(Tv,TSv) + \gamma\eta(Tu,Tv)$$

Theorem 1. Let (X,η) be a cone pentagonal metric space and P benormal cone with normal constant K. Let $T, S : X \rightarrow X$ be the twoself-mappings such that for all $u,v \in X$ satisfy the below:

$$\eta(TSu,TSv) \leq \alpha[\eta(Tu,TSu) + \eta(Tv,TSv)] \qquad (1)$$

where exists $\alpha \in [0,\frac{1}{2})$. Assume T is one-one and $T(X)$ is a completesubspace of X, then the mapping S has a unique fixed point in X.Furthermore, if S and T are commuting at the fixed point S, then $S$and T have a unique common fixed point in X.

Proof. Let $u_0 \in X$ be an arbitrary point. Take $u_{n+1} = Su_n$ be aniterative sequence in X for all $n = 0,1,2,...$Here one may noticethat if $u_{n+1} = u_n$ for any n implies that u_n become a fixed point of Sso always suppose that $u_{n+1} \neq u_n$ for all $n \geq 0$.Now from equation(1) we have,

$$\begin{aligned}\eta(Tu_n,Tu_{n+1}) &= \eta(TSu_{n-1},T\,Su_n)\\ &\leq \alpha[\eta(Tu_{n-1},TSu_{n-1}) + \eta(T\,u_n,T\,Su_n)]\\ &= \alpha[\eta(T\,u_{n-1},Tu_n) + \eta(T\,u_n,Tu_{n+1})]\end{aligned}$$

This implies,

$$\eta(Tu_n,Tu_{n+1}) \leq \frac{1}{1-\alpha}\,\eta(T\,u_{n-1},Tu_n)$$

for all $n \geq 0$. Hence,

$$\eta(Tu_n,Tu_{n+1}) \leq \nu\eta(T\,u_{n-1},Tu_n) \leq \nu^2\,\eta(T\,u_{n-2},Tu_{n-1}) \leq \ldots \leq \nu^n\,\eta(T\,u_0,T\,u_1), \qquad (2)$$

Where $\nu = \frac{1}{1-\alpha} < 1$.

ISSN: 2005-4289 IJDRC

We now show that u_0 is not a periodic point of S. If $u_0 = u_n$ for any $n \geq 2$, then from (2), we obtained

$$\eta(T\,u_0, T\,Su_0) = \eta(T\,u_n, T\,Su_n) = \eta(Tu_n, Tu_{n+1}) \leq \nu^n\,\eta(T\,u_0, T\,u_1)$$

Since $\nu < 1$, the above inequality gives a contradiction. Therefore always suppose that $u_n \neq u_m$ for all distinct $n, m \in \mathbb{N}$.

From (1), (2) and using the facts $\alpha \leq \nu$ and $0 \leq \alpha < \frac{1}{2} < 1$, we obtained

$$\begin{aligned}\eta(T\,u_n, T\,u_{n+2}) &= \eta(T\,Su_{n-1}, T\,Su_{n+1})\\ &\leq \alpha[\eta(T\,u_{n-1}, T\,Su_{n-1}) + \eta(T\,u_{n+1}, T\,Su_{n+1})]\\ &= \alpha[\eta(T\,u_{n-1}, Tu_n) + \eta(T\,u_{n+1}, T\,u_{n+2})]\\ &\leq \alpha[\nu^{n-1}\,\eta(T\,u_0, T\,u_1) + \nu^{n+1}\eta(T\,u_0, T\,u_1)]\\ &\leq \nu^n\,\eta(T\,u_0, T\,u_1) + \nu^{n+1}\eta(T\,u_0, T\,u_1) + \nu^{n+2}\eta(T\,u_0, T\,u_1)\\ &\leq (1 + \nu + \nu^2 + \dots)\nu^n\,\eta(T\,u_0, T\,u_1)\forall\, n \geq 0\end{aligned}$$

and

$$\begin{aligned}\eta(T\,u_n, T\,u_{n+3}) &= \eta(T\,Su_{n-1}, T\,Su_{n+2})\\ &\leq \alpha[\eta(T\,u_{n-1}, T\,Su_{n-1}) + \eta(T\,u_{n+2}, T\,Su_{n+2})]\\ &= \alpha[\eta(T\,u_{n-1}, Tu_n) + \eta(T\,u_{n+1}, T\,u_{n+3})]\\ &\leq \alpha[\nu^{n-1}\,\eta(T\,u_0, T\,u_1) + \nu^{n+2}\eta(T\,u_0, T\,u_1)]\\ &\leq \nu^n\,\eta(T\,u_0, T\,u_1) + \nu^{n+1}\eta(T\,u_0, T\,u_1) + \nu^{n+2}\eta(T\,u_0, T\,u_1)\\ &\leq (1 + \nu + \nu^2 + \dots)\nu^n\,\eta(T\,u_0, T\,u_1)\forall\, n \geq 0\end{aligned}$$

Now, if $m > 3$ and $m = 3k + 1$, where $k \geq 1$ from pentagonal property, we have,

$$\begin{aligned}\eta(T\,u_n, T\,u_{n+3k+1}) &\leq \eta(T\,u_{n+3k+1}, T\,u_{n+3k}) + \eta(T\,u_{n+3k}, Tu_{n+3k-1})\\ &\quad + \eta(T\,u_{n+3k-1}, T\,u_{n+3k-2}) + \eta(T\,u_{n+3k-2}, T\,u_n)\\ &\leq \eta(T\,u_{n+3k}, T\,u_{n+3k+1}) + \eta(T\,u_{n+3k-1}, T\,u_{n+3k})\\ &\quad + \eta(Tu_{n+3k-2}, T\,u_{n+3k-1}) + \eta(T\,u_{n+3k-2}, T\,u_{n+3k-3})\\ &\quad + \eta(T\,u_{n+3k-3}, T\,u_{n+3k-4}) + \dots + \eta(T\,u_{n+2}, T\,u_{n+1})\\ &\quad + \eta(T\,u_{n+1}, T\,u_n)\\ &= \eta(T\,u_n, T\,u_{n+1}) + \eta(T\,u_{n+1}, T\,u_{n+2}) + \cdots\\ &\quad + \eta(T\,u_{n+3k-1}, T\,u_{n+3k}) + \eta(T\,u_{n+3k}, T\,u_{n+3k+1})\\ &\leq \nu^n\,\eta(T\,u_0, T\,u_1) + \nu^{n+1}\eta(T\,u_0, T\,u_1) + \cdots\\ &\quad + \nu^{n+3k-1}\,\eta(T\,u_0, T\,u_1) + \nu^{n+3k}\,\eta(T\,u_0, T\,u_1)\\ &\leq \nu^n\,(1 + \nu + \nu^2 + \dots)\eta(T\,u_0, T\,u_1)\\ &= \frac{\nu^n}{1-\nu}\eta(T\,u_0, T\,u_1)\end{aligned}$$

In a similar manner, if $m > 4$ and $m = 3k + 2$, $k \geq 1$, then from pentagonal property, we have,

$$\begin{aligned}\eta(T\,u_n, T\,u_{n+3k+2}) &\leq \eta(T\,u_{n+3k+2}, T\,u_{n+3k+1}) + \eta(T\,u_{n+3k+1}, Tu_{n+3k})\\ &\quad + \eta(T\,u_{n+3k}, T\,u_{n+3k-1}) + \eta(T\,u_{n+3k-1}, T\,u_n)\\ &= \eta(T\,u_n, T\,u_{n+1}) + \eta(T\,u_{n+1}, T\,u_{n+2}) + \cdots\\ &\quad + \eta(T\,u_{n+3k-1}, T\,u_{n+3k}) + \eta(T\,u_{n+3k}, T\,u_{n+3k+1})\end{aligned}$$

ISSN: 2005-4289 IJDRC

$$\begin{aligned}&\leq \nu^n \eta(T u_0, T u_1) + \nu^{n+1}\eta(T u_0, T u_1) + \cdots \\ &\quad + \nu^{n+3k-1} \eta(T u_0, T u_1) + \nu^{n+3k} \eta(T u_0, T u_1) \\ &\leq \nu^n (1 + \nu + \nu^2 + \ldots)\eta(T u_0, T u_1) \\ &= \frac{\nu^n}{1-\nu}\eta(T u_0, T u_1)\end{aligned}$$

Again, if $m > 5$ and $m = 3k + 3$, $k \geq 1$, then from pentagonal property,

$$\begin{aligned}\eta(T u_n , T u_{n+3k+3}) &\leq \eta(T u_{n+3k+3}, T u_{n+3k+2}) + \eta(T u_{n+3k+2}, Tu_{n+3k+1}) \\ &\quad + \eta(T u_{n+3k+1}, T u_{n+3k}) + \eta(T u_{n+3k}, T u_n) \\ &\leq \eta(T u_{n+3k+1}, T u_{n+3k+2}) + \eta(T u_{n+3k}, T u_{n+3k+1}) \\ &\quad + \eta(Tu_{n+3k-1}, T u_{n+3k}) + \eta(T u_{n+3k-1}, T u_{n+3k-2}) \\ &\quad + \eta(T u_{n+3k-2}, T u_{n+3k-1}) + \ldots + \eta(T u_{n+2}, T u_{n+1}) \\ &\quad + \eta(T u_{n+1}, T u_n) \\ &= \eta(T u_n , T u_{n+1}) + \eta(T u_{n+1}, T u_{n+2}) + \cdots \\ &\quad + \eta(T u_{n+3k-1}, T u_{n+3k}) + \eta(T u_{n+3k}, T u_{n+3k+1}) \\ &\leq \nu^n \eta(T u_0, T u_1) + \nu^{n+1}\eta(T u_0, T u_1) + \cdots \\ &\quad + \nu^{n+3k-1} \eta(T u_0, T u_1) + \nu^{n+3k} \eta(T u_0, T u_1) \\ &\leq \nu^n (1 + \nu + \nu^2 + \ldots)\eta(T u_0, T u_1) \\ &= \frac{\nu^n}{1-\nu}\eta(T u_0, T u_1)\end{aligned}$$

Hence, by considering above cases, we obtain

$$\eta(T u_n, T u_{n+m}) \leq \frac{\nu^n}{1-\nu}\eta(T u_0, T u_1), \text{ for all } n, m \in \mathbb{N}.$$

Since P is a normal cone with normal constant K then we have,

$$\|\eta(T u_n, T u_{n+m}) \| \leq K \frac{\nu^n}{1-\nu} \|\eta(T u_0, T u_1) \|$$

for all $n, m \in \mathbb{N}$.

Since, $\lim_{n\to\infty} K \frac{\nu^n}{1-\nu} \|\eta(T u_0, T u_1) \| = 0$, we have,

$$\lim_{n\to\infty} \|\eta(T u_n, T u_{n+m}) \| = 0 \qquad (3)$$

for all $n, m \in \mathbb{N}$.

Thus, $\{Tu_n\}$ is a Cauchy sequence in X. Since $T(X)$ is a complete subspace of X, therefore there exists a point y in $T(X)$ such that,

$$\lim_{n\to\infty} Tu_{n+1} = \lim_{n\to\infty} TSu_n = y. \qquad (4)$$

Also, we can find a point $z \in X$ such that $y = T z$.
Now we show that$TSz = y$ By pentagonal property and (1), we have

$$\begin{aligned}\eta(y, TSz) &\leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2}) \\ &\quad + \eta(Tu_{n+2}, TSz) \\ &\leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2}) \\ &\quad + \eta(Tu_{n+1}, TSz)\end{aligned}$$

ISSN: 2005-4289 IJDRC

$$\leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2}) + \alpha[\eta(Tu_{n+1}, TSu_{n+1}) + \eta(Tz, T\,Sz)]$$

which implies that,

$$\eta(y, TSz) \leq \frac{1}{1-\alpha}[\eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + (1+\alpha)\eta(Tu_{n+1}, Tu_{n+2})]$$

Therefore,

$$\|\eta(y, TSz)\| \leq \frac{K}{1-\alpha}[\|\eta(y, Tu_n)\| + \|\eta(Tu_n, Tu_{n+1})\| + (1+\alpha)\|\eta(Tu_{n+1}, Tu_{n+2})\|]$$

As $n \to \infty$. We have $\|\eta(y, TSz)\| = 0$.

Hence, $TSz = Tz = y$. Since T is one to one, $z = Sz$. Thus, $z \in X$ is fixed point S. Now we show that z is unique fixed point of S.

Assume that x be another fixed point of S, that is, $x = Sx$. Then,

$$\begin{aligned}\eta(Tz, T\,x) &= \eta(TSz, T\,Sx)\\ &\leq \alpha[\eta(Tz, TSz) + \eta(Tx, TSx)]\\ &= \alpha[\eta(Tz, T\,z) + \eta(Tx, Tx)]\\ &= 0.\end{aligned}$$

Thus, $Tx = Tz$. Since T is one-one so that $x = z$. Since, S and T are commuting at the fixed point of S. Therefore Tz is a fixed point of S. Since we proved that S has a unique fixed point. Therefore $Tz = Sz = z$i.e., unique common fixed point of S and T is $z \in X$.

Corollary 2. [1] Let (X, η) be a cone pentagonal metric space and P be a normal cone with normal constant K. Let $T : X \to X$ be the self-mapping such that for all $u, v \in X$ satisfy the below:

$$\eta(T\,u, T\,v) \leq \alpha[\eta(T\,u, u) + \eta(T\,v, v)]$$

where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point in X.

Proof. On putting $S = I$, where I is the identity mapping on X, in theorem 1, the proof follows directly.

Theorem 3. Let (X, η) be a cone pentagonal metric space and P be a normal cone with normal constant K. Let $T\,, S : X \to X$ be the two self mappings such that for all $u, v \in X$ satisfy the below: $\eta(T\,Su, T\,Sv) \leq \lambda\eta(T\,u, T\,Su) + \beta\eta(T\,v, T\,Sv) + \gamma\eta(T\,u, T\,v)$(5) where $\lambda, \beta, \gamma \geq 0$ with $\lambda + \beta + \gamma < 1$. Assume T is one-one and $T\,(X)$ is a complete subspace of X, then the mapping S has a unique fixed point in X. Furthermore, if S and T are commuting at the fixed point S, then S and T have a unique common fixed point in X.

Proof. Let $u_0 \in X$ be an arbitrary point. Take $u_{n+1} = Su_n$ be an iterative sequence in X for all $n = 0, 1, 2, \ldots$ Here one may notice that if $u_{n+1} = u_n$ for any n implies that u_n become

ISSN: 2005-4289 IJDRC

a fixed point of S so always suppose that $u_{n+1} \neq u_n$ for all $n \geq 0$. Now from equation (5) we have

$$\eta(Tu_n, Tu_{n+1}) = \eta(TSu_{n_1}, TSu_n)$$
$$\leq \lambda\eta(Tu_{n-1}, TSu_{n_1}) + \beta\eta(Tu_n, TSu_n) + \gamma\eta(Tu_{n-1}, Tu_n)$$
$$= \lambda\eta(Tu_{n-1}, Tu_n) + \beta\eta(Tu_n, Tu_{n+1}) + \gamma\eta(Tu_{n-1}, Tu_n)$$

This implies,

$\eta(Tu_n, Tu_{n+1}) \leq \frac{\lambda+\gamma}{1-\beta}\eta(Tu_{n-1}, Tu_n)$for all $n \geq 0$. Hence,

$$\eta(Tu_n, Tu_{n+1}) \leq \mu\eta(Tu_{n-1}, Tu_n) \leq \mu^2\,\eta((Tu_{n-2}, Tu_{n-1}) \leq \ldots \leq \mu^n\eta(Tu_0, Tu_1), \quad (6)$$

where $\mu = \frac{\lambda+\gamma}{1-\beta} < 1$.

By using the similar discussion as theorem 1, we can prove that $\{Tu_n\}$ is a Cauchy sequence in X. Since $T(X)$ is a complete subspace of X, therefore there exists a point y in $T(X)$ such that,

$$\lim_{n\to\infty} Tu_{n+1} = \lim_{n\to\infty} TSu_n = y \quad (7)$$

Also, we can find a point $z \in X$ such that $y = T\,z$. Now we show that $TSz = y$. By pentagonal property, we have

$$\eta(y, TSz) \leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2}) + \eta(Tu_{n+2}, TSz)$$
$$\leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2}) + \eta(TSu_{n+1}, TSz)$$
$$\leq \eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + \eta(Tu_{n+1}, Tu_{n+2})$$
$$+ [\lambda\eta(Tu_{n+1}, TSu_{n+1}) + \beta\eta(Tz, TSz) + \gamma\eta(Tu_n, y)]$$

which implies that,

$$\eta(y, TSz) \leq \frac{1}{1-\beta}\,[(1+\gamma)\eta(y, Tu_n) + \eta(Tu_n, Tu_{n+1}) + (1+\lambda)\eta(Tu_{n+1}, Tu_{n+2})]$$

Therefore,

$$\|\,\eta(y, TSz)\| \leq \frac{K}{1-\beta}\,[(1+\gamma)\|\eta(y, Tu_n)\| + \|\eta(Tu_n, Tu_{n+1})\| + (1$$
$$+ \lambda)\|\eta(Tu_{n+1}, Tu_{n+2})\|]$$

As $n \to \infty$. We have $\|\eta(y, TSz)\| = 0$.

$$\eta(Tz, Tx) = \eta(TSz, TSx)$$
$$\leq \lambda\eta(Tz, TSz) + \beta\eta(Tx, TSx) + \gamma\eta(Tz, Tx)]$$
$$\leq \lambda\eta(Tz, Tz) + \beta\eta(Tx, Tx) + \gamma\eta(Tz, Tx)]$$
$$\leq \gamma\eta(Tz, Tx)$$

As $\gamma < 1$, then $\eta(Tz, Tx) = 0$. Thus, $Tx = Tz$. Since T is one-one so that $x = z$. Since, S and T are commuting at the fixed point of S. ThereforeTz is a fixed point of S. Since we

ISSN: 2005-4289 IJDRC

proved that S has a unique fixed point. Therefore $Tz = Sz = z$ i.e., unique common fixed point of S and T is $z \in X$.

ACKNOWLEDGEMENT

The authors present their obligation to the referees for their great work.

REFERENCES

1. Auwalu A., Kannan fixed point theorem in a cone pentagonalmetric spaces, J. Math. Comput. Sci., 6, (2016), 515-526.
2. Azam A., Arshad M., and Beg I., Banach contraction principleon cone rectangular metric spaces, Appl. Anal. Discrete Math.,3, (2009), 236-241.
3. Garg M. and Agarwal S., Banach Contraction Principle on Cone Pentagonal Metric Space, Journal of Advanced Studies in Topology, 3, (2012), 12-18.
4. Huang and Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332, (2007), 1468-1476.
5. Kannan R., Some results on fixed point-II, Amer. Math. Monthly,76, (1969), 405-408.
6. Abbas M. and Jungck G., Common fixed point results for non-commuting mappings without continuity in Cone metric spaces,J. Math. Anal. Appl., 341, (2008), 416-420.
7. Jleli M. and Samet B., The Kannan fixed point theorem in acone rectangular metric space, J. Nonlinear Sci. Appl., 2, (2009), 161-167.
8. Malhotra S. K., Shukla S., and Sen R., Some coincidence and common fixed point theorems in cone metric spaces, Bulletin of Mathematical Analysis and Applications, 4, (2012), 64-71.
9. Morales J. and Rojas E., Cone metric spaces and fixed point theorems of T-Kannan contractive mappings, Int. Journal of Math.Analysis, 4, (2010), 175-184.

ISSN: 2005-4289 IJDRC