International Journal of Control and Automation
Vol. 12, No. 4, (2019), pp. 229-239

Piecewise weighted pseudo almost periodic solutions for two-term time-fractional impulsive differential equations

Lalit Kumar Som[a], Chaman Singh[b,*]

[a]*Department of Mathematics, Motilal Nehru College, New Delhi*-110021
[b]*Department of Mathematics, Choudhary Charan Singh University Meerut, Meerut*-250004

Abstract

In this paper, we develop the sufficient conditions for the existence and uniqueness of piecewise weighted pseudo almost periodic mild solutions for two-term time-fractional impulsive differential equations in Banach space. The working tools are based on sectorial operator, fixed point techniques and generalization of the semigroup theory of linear operators. An example is given to illustrate the theory.

Keywords:
Two-term time-fractional derivative, sectorial operator, piecewise weighted pseudo almost periodic functions.
2010 MSC: 22D40, 34A08, 34G20, 35R11, 35R12, 43A60.

1. Introduction

In this paper, we are interested to investigate the existence and uniqueness of piecewise weighted pseudo almost periodic mild solutions for the following two-term time-fractional impulsive differential equations

$$^cD^{\mu+1}y(t) + \beta^cD^{\nu}y(t) = Ay(t) +^c D^{\mu}f(t, y(t)), \quad t > 0, t \neq t_k, \tag{1.1}$$

$$\Delta y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1, 2, 3, ..., \tag{1.2}$$

$$y(0) = y'(0) = 0. \tag{1.3}$$

where $0 < \mu \leq \nu \leq 1, \beta > 0$, $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is closed linear operator and $^cD^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha > 0$; f, I_k are given functions satisfying some appropriate conditions which will be mentined later. $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(\cdot)$ at t_k, $k \in \mathbb{Z}^+$, respectively.

In the recent years, the study of fractional differential equations have been gaining considerable attention and increasing interest of many scientists and mathematicians in different context and areas of research due to the fact that fractional order derivative provide a tool for description of memory and hereditary properties of various phenomena. For more detail on the topic of fractional order differential equations see the papers [6, 9, 12, 16, 21, 22], the monographs [3, 4, 20] and references therein.

On the other hand, the study of differential equations with impulsive effects constitute a useful and interesting field of research due to a lot of applications in various fields such as chemical engineering, physics, medicine and economics etc. The fractional differential equations involving impulsive effects came out as a natural description of observed phenomena. For more detail see [9, 17, 26, 27].

The system $(1.1) - (1.3)$ is a general model that includes recent investigations in this subject. Indeed, in [13, 21, 25, 28] the authors have obtained the existence and uniqueness results without impulsive

ISSN: 2005-4297 IJCA

conditions and, in [21], Pardo studied weighted pseudo almost automorphic mild solutions for two-term time-fractional order differential equations.

In particular, the concept of existence and uniqueness of piecewise weighted pseudo almost periodic solutions introduced by Xia [26] is one of the most attractive topic in qualitative theory of differential equations. The generalizations of classical almost periodic functions and their plentiful applications in differential equations with impulsive effects have been studied by numerous authors (see [9, 12, 17, 24, 26, 27, 29]). Anticipating a great interest in the problems modeled as the system (1.1) − (1.3), this paper contributes to fill this important gap.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, $\Omega \subset \mathbb{X}$ and $\mathbb{K}$ be a compact subset of Ω. Let $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}$ and $\mathbb{Z}^+$ stands for real number, nonnegative real numbers, integers and nonnegative integers respectively. For a linear operator A on $\mathbb{X}$, $\mathcal{R}(A), D(A)$ and $\varrho(A)$ represent range, domain and resolvent of A respectively. Let $\mathcal{T}$ be the collections of all sequences $\{t_k\}_{k\in\mathbb{Z}^+}$ such that $\kappa = \inf_{k\in\mathbb{Z}^+}(t_{k+1} - t_k) > 0$. For brevity, we introduce the following notations:

- $\mathcal{C}(\mathbb{R}, \mathbb{X})$: the space of all functions $g : \mathbb{R} \to \mathbb{X}$ which are continuous.
- $\mathcal{PC}(\mathbb{R}, \mathbb{X})$: the space of all functions $g : \mathbb{R} \to \mathbb{X}$ which are piecewise continuous such that $g(\cdot)$ is continuous at t for $t \notin \{t_k\}_{k\in\mathbb{Z}}$, $g(t_k^+), g(t_k^-)$ exists and $g(t_k^-) = g(t_k)$ for all $k \in \mathbb{Z}$.
- $\mathcal{PC}(\mathbb{R} \times \Omega, \mathbb{X})$: the space of all piecewise continuous functions $g : \mathbb{R} \times \Omega \to \mathbb{X}$ such that for any $y \in \Omega$, $g(\cdot, y) \in \mathcal{PC}(\mathbb{R}, \mathbb{X})$ and for any $t \in \mathbb{R}$, $g(t, \cdot)$ is continuous at $y \in \Omega$.

Now, we recall some definitions and basic results on fraction calculus (for more details, see [23]). Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0; \\ 0, & t \le 0, \end{cases}$$

where Γ denote gamma function. The function g_η has the properties $(g_a * g_b)(t) = g_{a+b}(t)$, for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and Re $\eta > 0$, where $\widehat{(\cdot)}$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively.

Definition 2.1. *Riemann-Liouville fractional integral of a locally integrable function $f : \mathbb{R}^+ \mapsto \mathbb{X}$ of order $\eta > 0$ is defined as follows*

$$J_t^\eta f(t) = (g_\eta * f)(t) = \int_0^t g_\eta(t-s) f(s) ds, \quad t > 0,$$

and $J_t^0 f(t) := f(t)$.

Definition 2.2. *The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m(\mathbb{R}^+, \mathbb{E})$ with lower limit 0 is given by*

$${}^cD_t^\eta f(t) = I_t^{m-\eta} D_t^m f(t) = \int_0^t g_{m-\eta}(t-s) \frac{d^m}{dt^m} f(s) ds, \quad m-1 < \eta \le m.$$

Definition 2.3. *A densely defined closed linear operator A is said to be sectorial of type ω and angle θ if there exists $\theta \in [0, \frac{\pi}{2})$, $\mathcal{M} > 0$, $\omega \in \mathbb{R}$, such that its resolvent exists in the sector*

$$\begin{aligned} \omega + \Sigma_\theta :=& \{\omega + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\} \backslash \{\omega\}, \\ \|(\lambda I - A)^{-1}\| \le& \frac{\mathcal{M}}{|\lambda - \omega|}, \quad \lambda \in \omega + \Sigma_\theta. \end{aligned} \tag{2.1}$$

ISSN: 2005-4297 IJCA

We Note that A is a sectorial of angle $\frac{\pi}{2}+\theta$ if $\omega=0$. Generally, (2.1) does not holds in a sector of angle $\frac{\pi}{2}$. Our restriction corresponds to the class of operators used in this paper.

Definition 2.4. *[13] Let $\beta>0$ and $0\leq\mu,\nu\leq 1$ be given. Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $D(A)$. Then A is said to be the generator of a $(\mu,\nu)_\beta-$ regularized family if there exists $\omega\geq 0$ and a strongly continuous function $\mathcal{S}_{\mu,\nu}:\mathbb{R}^+\to\mathcal{B}(\mathbb{X})$(the space of all bounded operator on $\mathbb{X}$) such that $\{\lambda^{\mu+1}+\beta\lambda^\nu : Re\,\lambda>\omega\}\subset\varrho(A)$ and*

$$\lambda^\mu\bigg(\lambda^{\mu+1}+\beta\lambda^\nu-A\bigg)^{-1}y=\int_0^\infty e^{-\lambda t}\mathcal{S}_{\mu,\nu}(t)ydt,\quad Re\,\lambda>\omega, y\in\mathbb{X}.$$

By the Laplace transform, we know that $\beta=0$ and $\mu=1$ corresponds to the concept of cosine family whereas $\beta=0$ and $\mu=0$ corresponds to the case of a C_0- semigroup. We refer to the monograph [2] for more information about Laplace approach to semigroup and cosine families.

Theorem 2.5. *[13] Let $\beta>0,\omega<0$ and $0<\mu\leq\nu\leq 1$ be given. Suppose that A is an ω-sectorial operator of angle $\frac{\nu\pi}{2}$, then A generates an $(\mu,\nu)_\beta-$ regularized family $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0}$ satisfying the estimate*

$$\|\mathcal{S}_{\mu,\nu}(t)\|\leq\frac{M}{1+|\omega|(t^{\mu+1}+\beta t^\nu)},\quad t\geq 0, \tag{2.2}$$

where M is the constant depending solely on μ,ν.

Note that

$$\int_0^\infty\frac{1}{1+|\omega|t^{\mu+1}}dt=\frac{|\omega|^{-1/(\mu+1)}\pi}{(\mu+1)\sin(\pi/(\mu+1))}, \tag{2.3}$$

thus $\mathcal{S}_{\mu,\nu}(t)$ is integrable on $(0,\infty)$ for $0<\mu\leq 1$.

Next, we recall the basic concepts of piecewise weighted pseudo almost periodic solutions [26, 27, 28, 29].

Definition 2.6. *A function $y\in\mathcal{C}(\mathbb{R},\mathbb{X})$ is said to be almost periodic if for ever $\epsilon>0$, there exists a $l(\epsilon)>0$ such that every interval $[a,a+l]$, $a\in\mathbb{R}$ contains a τ satisfying*

$$\|y(t+\tau)-y(t)\|<\epsilon,$$

for all $t\in\mathbb{R}$ with the property $\mathbb{R}\cap[a,a+l]\neq\emptyset$.

Definition 2.7. *A sequence $\{z_n\}$ is said to be almost periodic sequence if for ever $\epsilon>0$, there exists $l(\epsilon)>0$ such that the interval $[p,p+l]$, $p\in\mathbb{Z}$ contains at least one number k satisfying the property $\|z_{n+k}-z_n\|<\epsilon$ for all $n\in\mathbb{Z}$ with the property $\mathbb{Z}\cap[p,p+l]\neq\emptyset$. We denote the set of all such functions by $\mathcal{AP}(\mathbb{Z},\mathbb{X})$.*

Let W_d denotes the set of all functions (weights) $\rho:\mathbb{Z}^+\to(0,+\infty)$. For $\rho\in W_d$ and $m\in\mathbb{Z}^+$, set $w(m,\rho):=\sum_{k=0}^m\rho_k$. Denote $W_{d,\infty}:=\{\rho\in W_d:\lim_{m\to\infty}w(m,\rho)=\infty\}$.

For $\rho\in W_{d,\infty}$, we define

$$\mathcal{PC}_0(\mathbb{Z}^+,\mathbb{X})=\bigg\{z_n\in l^\infty(\mathbb{Z}^+,\mathbb{X}):\lim_{n\to\infty}\|z_n\|=0\bigg\}, \tag{2.4}$$

$$\mathcal{PAP}_\rho(\mathbb{Z}^+,\mathbb{X})=\bigg\{z_n\in l^\infty(\mathbb{Z}^+,\mathbb{X}):\lim_{m\to\infty}\frac{1}{w(m,\rho)}\sum_{k=0}^m\|z_k\|\rho_k=0\bigg\}. \tag{2.5}$$

Definition 2.8. *We call a sequence $\{z_n\}_{n\in\mathbb{Z}^+}\in l^\infty(\mathbb{Z}^+,\mathbb{X})$ discrete asymptotically almost periodic if $z_n=a_n+b_n$, where $a_n\in\mathcal{AP}(\mathbb{Z},\mathbb{X})$ and $b_n\in\mathcal{PC}_0(\mathbb{Z}^+,\mathbb{X})$. We denote the set of all such sequences by $\mathcal{AAP}_0(\mathbb{Z}^+,\mathbb{X})$.*

Definition 2.9. *Let $\rho \in W_{d,\infty}$. A sequence $\{z_n\}_{n\in\mathbb{Z}^+} \in l^{\infty}(\mathbb{Z}^+, \mathbb{X})$ is called weighted pseudo almost periodic if $z_n = a_n + b_n$, where $a_n \in \mathcal{AP}(\mathbb{Z}, \mathbb{X})$ and $b_n \in \mathcal{PAP}_{\rho}(\mathbb{Z}^+, \mathbb{X})$. The set of all such functions denoted by $\mathcal{WPAP}_{\rho}(\mathbb{Z}^+, \mathbb{X})$.*

Definition 2.10. *We call a function $y \in \mathcal{PC}(\mathbb{R}, \mathbb{X})$ piecewise almost periodic if:*

(i) *$\{t_k^j : t_k^j = t_{k+j} - t_k, k, j \in \mathbb{Z}\}$ is equipotentially almost periodic i.e. for every $\epsilon > 0$ and for all sequences there exists a relative dense set in $\mathbb{R}$ with common ϵ-almost periods.*

(ii) *For every points s and t contained in the same interval of continuity of $y(t)$ for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|t - s| < \delta$, then $\|y(t) - y(s)\| < \epsilon$.*

(iii) *For ever $\epsilon > 0$, there exists a $l(\epsilon) > 0$ such that every interval $[a, a+l]$, $a \in \mathbb{R}$ contains a τ satisfying*

$$\|y(t + \tau) - y(t)\| < \epsilon,$$

for all $t \in \mathbb{R}$, $|t - t_k| > \epsilon, k \in \mathbb{Z}$ with the property $\mathbb{R} \cap [a, a + l] \neq \emptyset$.

We symbolize the set of all piecewise almost periodic functions by $\mathcal{AP}(\mathbb{R}, \mathbb{X})$. We denote by $\mathcal{UPC}(\mathbb{R}, \mathbb{X})$ the set of all functions satisfy the condition (ii) in Definition 2.10.

Definition 2.11. *[5] A function $f(t, y) \in \mathcal{PC}(\mathbb{R} \times \Omega, \mathbb{X})$ is called piecewise almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \Omega$, if for every given $\epsilon > 0$, there exists a relatively dense set R_ϵ of $\mathbb{R}$, such that $\|f(t + \tau, y) - f(t, y)\| < \epsilon$, for all $y \in \mathbb{K}, \tau \in R_\epsilon$ and $t \in \mathbb{R}$ with $|t - t_k| > \epsilon, k \in \mathbb{Z}$ and $\{f(\cdot, y) : y \in \mathbb{K}\}$ is uniformly bounded. The set of all such functions is denoted by $\mathcal{AP}(\mathbb{R} \times \Omega, \mathbb{X})$.*

Lemma 2.12. *[8] Let $\{z_k\}_{k\in\mathbb{Z}} \in \mathcal{AP}(\mathbb{Z}, \mathbb{X})$, $f \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$, and $\{\tau_k^j : k, j \in \mathbb{Z}\}$ be equipotentially almost periodic. Then for each $\epsilon > 0$ there exist relatively dense sets Z_ϵ of $\mathbb{Z}$ and R_ϵ of $\mathbb{R}$ such that the following conditions hold:*

(i) *$\|f(t + \tau) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$, $|t - t_k| > \epsilon$, $\tau \in R_\epsilon$ and $k \in \mathbb{Z}$.*

(ii) *$\|z_{k+p} - z_k\| < \epsilon$ for all $p \in Z_\epsilon$, and $k \in \mathbb{Z}$.*

(iii) *For any $\tau \in R_\epsilon$ there exists at least a number $p \in Z_\epsilon$ such that $|t_k^p - \tau| < \epsilon$, $k \in \mathbb{Z}$.*

Now, we recall the concept of piecewise weighted pseudo almost periodic functions and explore its properties.
Let W be the collections of all positive and locally integrable functions $\rho : \mathbb{R}^+ \to (0, \infty)$. For each $\rho \in W$ and $\gamma > 0$, set

$$w(\gamma, \rho) := \int_0^{\gamma} \rho(t)dt.$$

Define

$$W_\infty = \{\rho \in W : \lim_{\gamma\to\infty} w(\gamma, \rho) = \infty\},$$
$$W_B := \{\rho \in W_\infty : \rho \text{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $W_B \subset W_\infty \subset W$.

Definition 2.13. *Let $\rho_1, \rho_2 \in W_\infty$. Then ρ_1 and ρ_2 are said to be equivalent i.e. ($\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in W_B$.*

It is clear that " $\sim$ " binary equivalence relation on W_∞. For a given weight $\rho \in W_\infty$, the equivalence class is denoted by $C_L(\rho) := \{\rho* \in W_\infty : \rho \sim \rho*\}$. Moreover $W_\infty = \cup_{\rho\in W_\infty} C_L(\rho)$.
For $\rho \in W_\infty$, we define

$$\mathcal{PC}_0(\mathbb{R}^+, \mathbb{X}) = \left\{ f \in \mathcal{PC}(\mathbb{R}^+, \mathbb{X}) : \lim_{t\to\infty} \|f(t)\| = 0 \right\},$$
$$\mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X}) := \left\{ f \in \mathcal{PC}(\mathbb{R}^+, \mathbb{X}) : \lim_{\gamma\to\infty} \frac{1}{w(\gamma, \rho)} \int_0^{\gamma} \|f(t)\|\rho(t)dt = 0 \right\}.$$

ISSN: 2005-4297 IJCA

Similarly

$$\mathcal{PAP}_\rho(\mathbb{R}^+\times\Omega,\mathbb{X})$$
$$:=\left\{f\in\mathcal{PC}(\mathbb{R}^+\times\Omega,\mathbb{X}):\lim_{\gamma\to\infty}\frac{1}{w(\gamma,\rho)}\int_0^\gamma\|f(t,y)\|\rho(t)dt=0,\text{ uniformly in } y\in\mathbb{K}\right\}.$$

Definition 2.14. *A function $f\in\mathcal{PC}(\mathbb{R}^+,\mathbb{X})$ is said to be asymptotically almost periodic if $f=\phi+\psi$, where $\phi\in\mathcal{AP}(\mathbb{R},\mathbb{X})$ and $\psi\in\mathcal{PC}_0(\mathbb{R}^+,\mathbb{X})$. We denote all such functions by $\mathcal{AAP}_0(\mathbb{R}^+,\mathbb{X})$.*

Definition 2.15. *A function $f\in\mathcal{PC}(\mathbb{R}^+,\mathbb{X})$ is called piecewise weighted pseudo almost periodic if it has a decomposition of the form $f=\phi+\psi$, where $\phi\in\mathcal{AP}(\mathbb{R},\mathbb{X})$ and $\psi\in\mathcal{PAP}_\rho(\mathbb{R}^+,\mathbb{X})$. The set of all such functions denoted by $\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})$.*

Definition 2.16. *A function $f\in\mathcal{PC}(\mathbb{R}^+\times\Omega,\mathbb{X})$ is called piecewise weighted pseudo almost periodic if it has a decomposition of the form $f=\phi+\psi$, where $\phi\in\mathcal{AP}(\mathbb{R}\times\Omega,\mathbb{X})$ and $\psi\in\mathcal{PAP}_\rho(\mathbb{R}^+\times\Omega,\mathbb{X})$. The set of all such functions denoted by $\mathcal{WPAP}_\rho(\mathbb{R}^+\times\Omega,\mathbb{X})$.*

For $\rho\in W_\infty$ and $\tau\in\mathbb{R}^+$ define ρ^τ by $\rho^\tau(t)=\rho(t+\tau)$ for all $t\in\mathbb{R}^+$. Define

$$W_T=\{\rho\in W_\infty:\rho(t)\sim\rho^\tau(t)\text{ for each } t\in\mathbb{R}^+\}.$$

It is clear that W_T contains many of weights, such as $1,e^t$ and $1+|t|^n$ with $n\in\mathbb{Z}^+$.

Remark 2.17. (i) *For $\rho\in W_T$, $\mathcal{PAP}_\rho(\mathbb{R}^+,\mathbb{X})$ is a translation invariant subset of $\mathcal{PC}_0(\mathbb{R}^+,\mathbb{X})$.*

(ii) *$\mathcal{PC}_0(\mathbb{R}^+,\mathbb{X})\subset\mathcal{PAP}_\rho(\mathbb{R}^+,\mathbb{X})$ and $\mathcal{AAP}_0(\mathbb{R}^+,\mathbb{X})\subset\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})$.*

(i) *It is easy to see that $\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})$(resp., $\mathcal{WPAP}_\rho(\mathbb{R}^+\times\Omega,\mathbb{X})$), $\rho\in W_T$ is a Banach spaces with sup norm norm.*

Similar as in [1] we have the following lemma.

Lemma 2.18. *Let $\{z_k\}_{k\in\mathbb{Z}^+}\in\mathcal{PAP}_\rho(\mathbb{Z}^+,\mathbb{X})$. Then there exists a function $g\in\mathcal{PAP}_\rho(\mathbb{R}^+,\mathbb{X})$ such that $g(k)=z_k$, $k\in\mathbb{Z}^+$.*

Lemma 2.19. *[8] Assume that $\{t_k^j:k,j\in\mathbb{Z}\}$ are equipotentially almost periodic sequences, then for each $p>0$ there exists a integer $N>0$ such that each interval of length p has no more than N elements of the sequence $\{t_k\}$ and*

$$i(s,t)\le N(t-s)+N,$$

where $i(t,s)$ represents the number of the points t_k in the interval $[t,s]$.

Similarly as the proof of [17] the following theorems hold for piecewise weighted pseudo almost periodic functions.

Theorem 2.20. *Let $f(t,y)\in\mathcal{WPAP}_\rho(\mathbb{R}^+\times\Omega,\mathbb{X})$. We consider that there exists $L_f>0$ such that*

$$\|f(t,y_1)-f(t,y_2)\|\le L_f.\|y_1-y_2\|,\quad\forall\ t\in\mathbb{R}^+,\ y_1,y_2\in\Omega.$$

If $\phi\in\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})$ such that $\mathcal{R}(\phi)\subset\Omega$, then $f(\cdot,\phi(\cdot))\in\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})$.

Theorem 2.21. *Let $\{I_k(y):k\in\mathbb{Z}^+\}$ for any $y\in\Omega$ is a weighted pseudo almost periodic sequence. We consider that there exists $L_0>0$ such that*

$$\|I_k(x)-I_k(y)\|\le L_0.\|x-y\|,\ \textit{for all } x,y\in\Omega,k\in\mathbb{Z}^+.$$

If $\phi\in\mathcal{WPAP}_\rho(\mathbb{R}^+,\mathbb{X})\cap\mathcal{UPC}(\mathbb{R}^+,\mathbb{X})$ satisfying $\mathcal{R}(\phi)\subset\Omega$, then $I_k(\phi(t_k))$ is weighted pseudo almost periodic.

3. Main Results

Now, we are interested to investigate the existence and uniqueness of piecewise weighted pseudo almost periodic mild solutions for the two-term time-fractional impulsive differential system (1.1)-(1.3).
In formulation of the system (1.1)-(1.3), we assume the following conditions:

(H_1) The collection of all sequences $\{\tau_k^j : k, j \in \mathbb{Z}\}$ is equipotentially almost periodic and there exists $\delta > 0$ such that $\inf_{k\in\mathbb{Z}} \tau_k^1 = \delta$.

(H_2) A is an ω sectorial operator of angle $\nu\pi/2$ with $\omega < 0$.

(H_3) Assume that $I_k \in \mathcal{WPAP}_\rho(\mathbb{Z}^+, \mathbb{X})$ and there exists a constant $L_I > 0$ such that

$$\|I_k(x) - I_k(y)\| \le L_I.\|x - y\|, \text{ for all } x, y \in \mathbb{X},\, k \in \mathbb{Z}^+.$$

(H_4) Assume that $f \in \mathcal{WPAP}_\rho(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$, $\rho \in W_T$ and there exists a $L_f > 0$ such that

$$\|f(t, x) - f(t, y)\| \le L_f\|x - y\|, \quad x, y \in \mathbb{X}, t \in \mathbb{R}^+.$$

Now, using the definition of $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\ge0}$ and Laplace transformation, we establish the mild solution of the system (1.1)-(1.3).

Definition 3.1. *Let $0 < \mu \le \nu \le 1, \beta > 0$. Assume that A is a ω-sectorial operator of angle $\frac{\nu\pi}{2}$. Then a function $y \in \mathcal{PC}(\mathbb{R}^+, \mathbb{X})$ is called a mild solution of the system (1.1)-(1.3) and given by*

$$y(t) = \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t - t_k)I_k(y(t_k)) + \int_0^t \mathcal{S}_{\mu,\nu}(t - s)f(s, y(s))ds. \tag{3.1}$$

Lemma 3.2. *Let $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\ge0} \subset \mathcal{B}(\mathbb{X})$ be a uniformly integrable and strongly continuous family. Assume that $h \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$, then*

$$(\Phi h)(t) := \int_0^t \mathcal{S}_{\mu,\nu}(t - s)h(s)ds \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X}).$$

Proof. First we show that (Φh) is well defined. Since $h \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$, then $\|h\| = \sup_{t\in\mathbb{R}^+} \|h(t)\| < \infty$. Now by the estimate (2.2) we have

$$\|\mathcal{S}_{\mu,\nu}(t - s)h(s)\| \le \frac{M}{1 + |\omega|[(t - s)^{\mu+1} + \beta(t - s)^\nu]}\|h\| \le \frac{M}{1 + |\omega|[(t - s)^{\mu+1}]}\|h\|,$$

then

$$\int_0^\infty \|\mathcal{S}_{\mu,\nu}(s)h(t - s)\|ds \le \frac{|\omega|^{-1/(\mu+1)}\pi}{(\mu + 1)\sin(\pi/(\mu + 1))}\|h\|.$$

Thus $\mathcal{S}_{\mu,\nu}(s)h(t - s)$ is integrable on $(0, \infty)$ for $0 < \mu \le \nu \le 1$.
In fact for $h = \phi + \psi$, where $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$ and $\psi \in \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$ we have that

$$\int_0^t \mathcal{S}_{\mu,\nu}(t - s)h(s)ds = \Phi_1(t) + \Phi_2(t)$$

where

$$\Phi_1(t) = \int_{-\infty}^t \mathcal{S}_{\mu,\nu}(t - s)\phi(s)ds, \text{ and } \Phi_2(t) = \int_0^t \mathcal{S}_{\mu,\nu}(t - s)\psi(s)ds - \int_{-\infty}^0 \mathcal{S}_{\mu,\nu}(t - s)\phi(s)ds.$$

ISSN: 2005-4297 IJCA

Similar as the proof in [26], we can show that $\Phi_1 \in \mathcal{UPC}(\mathbb{R}, \mathbb{X})$. Now, we show that $\Phi_1 \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$. Since $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$, there exists a relatively dense set R_ϵ such that for $\tau \in R_\epsilon$, $t \in \mathbb{R}$, $|t-t_k| > \epsilon$, $k \in \mathbb{Z}$,

$$\|\phi(t+\tau) - \phi(t)\| < \epsilon.$$

Hence for $t \in \mathbb{R}$, $|t-t_k| > \epsilon$, $k \in \mathbb{Z}$, we have

$$\begin{aligned}\|\Phi_1(t+\tau) - \Phi_1(t)\| =&\leq \int_{-\infty}^{t} \|\mathcal{S}_{\mu,\nu}(t-s)\| \|\phi(\tau+s) - \phi(s)\| \\ &< \epsilon \int_0^\infty \frac{M}{1+|\omega|(s^{\mu+1})} ds \\ &\leq \epsilon \frac{|\omega|^{-1/(\mu+1)}\pi}{(\mu+1)\sin(\pi/(\mu+1))},\end{aligned}$$

that is $\Phi_1 \in \mathcal{AP}(\mathbb{R}, \mathbb{X})$.

Next, we show that $\Phi_2 \in \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$. Note that the function $\Theta_\gamma(s) := \frac{1}{w(\gamma,\rho)} \int_0^{\gamma-s} \|\psi(\xi)\|\rho(\xi)d\xi$ is decreasing on $\mathbb{R}^+$. Moreover $\Theta_\gamma(0) = \frac{1}{w(\gamma,\rho)} \int_0^\gamma \|\psi(\xi)\|\rho(\xi)d\xi \to 0$ as $\gamma \to \infty$ by hypothesis. Further by Fubini's theorem we have

$$\begin{aligned}&\frac{1}{w(\gamma,\rho)} \int_0^\gamma \left\| \int_0^\xi \mathcal{S}_{\mu,\nu}(\xi-s)\psi(s)ds \right\| \rho(\xi)d\xi \\ &\leq \frac{1}{w(\gamma,\rho)} \int_0^\gamma \left[\int_0^\xi \|\mathcal{S}_{\mu,\nu}(\xi-s)\| \|\psi(s)\| ds \right] \rho(\xi)d\xi \\ &= \frac{1}{w(\gamma,\rho)} \int_0^\gamma \left[\int_s^\gamma \|\mathcal{S}_{\mu,\nu}(s)\| \|\psi(\xi-s)\| \rho(\xi)d\xi \right] ds \\ &= \int_0^\gamma \|\mathcal{S}_{\mu,\nu}(s)\| \left[\frac{1}{w(\gamma,\rho)} \int_s^\gamma \|\psi(\xi-s)\|\rho(\xi)d\xi \right] ds \\ &= \int_0^\gamma \|\mathcal{S}_{\mu,\nu}(s)\| \left[\frac{1}{w(\gamma,\rho)} \int_0^{\gamma-s} \|\psi(\xi)\|\rho(\xi-s)d\xi \right] ds \\ &= \int_0^\gamma \|\mathcal{S}_{\mu,\nu}(s)\| \Theta_\gamma(s) ds \\ &\leq \int_0^\infty \|\mathcal{S}_{\mu,\nu}(s)\| \Theta_\gamma(0) ds \to 0, \text{ (as } \gamma \to \infty).\end{aligned}$$

Hence $\int_0^t \mathcal{S}_{\mu,\nu}(t-s)\psi(s)ds \in \mathcal{PC}_0(\mathbb{R}^+, \mathbb{X}) \subset \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$. Further

$$\begin{aligned}\left\| \int_{-\infty}^0 \mathcal{S}_{\mu,\nu}(t-s)\phi(s)ds \right\| &\leq \int_{-\infty}^0 \|\mathcal{S}_{\mu,\nu}(t-s)\| \|\phi(s)\| ds \\ &\leq \|\phi\| \int_t^\infty \|\mathcal{S}_{\mu,\nu}(s)\| ds \to 0, \text{ (as } t \to \infty).\end{aligned}$$

This implies that $\int_{-\infty}^0 \mathcal{S}_{\mu,\nu}(t-s)\phi(s)ds \in \mathcal{PC}_0(\mathbb{R}^+, \mathbb{X}) \subset \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$. Thus from the above estimates we have $\Phi_2 \in \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$. □

Lemma 3.3. *Let $\{\mathcal{S}_{\mu,\nu}(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{X})$ be a uniformly integrable and strongly continuous family. Assume that $\{\alpha_k\}_{k\in\mathbb{Z}^+} \in \mathcal{WPAP}_\rho(\mathbb{Z}^+, \mathbb{X})$, then*

$$\Psi(t) := \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)\alpha_k \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X}).$$

Proof. Since $\{\alpha_k\}_{k\in\mathbb{Z}^+} \in \mathcal{WPAP}_\rho(\mathbb{Z}^+,\mathbb{X})$, then let $\alpha_k = a_k + b_k$, with a_k is an almost periodic sequence and $b_k \in \mathcal{PAP}_\rho(\mathbb{Z}^+,\mathbb{X})$, so

$$\sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)\alpha_k = \Psi_1(t) + \Psi_2(t),$$

where $\Psi_1(t) = \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)a_k$ and $\Psi_2(t) = \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)b_k$. Since $\{t_k^j : k,j \in \mathbb{Z}^+\}$ are equipotentially almost periodic, then by Lemma 2.12, for any $\epsilon > 0$, there exists relative dense set Z_ϵ of integers and R_ϵ of real numbers such that for $\tau \in R_\epsilon$, $q \in Z_\epsilon$ $t_k < t \leq t_{k+1}$, $|t-t_k| > \epsilon$, $|t-t_{k+1}| > \epsilon$, $k \in \mathbb{Z}^+$, we have

$$\begin{aligned} t+\tau > t_k + \tau + \epsilon > t_{k+q}, \\ t_{k+q+1} > t_{k+1} + \tau - \epsilon > t+\tau, \end{aligned}$$

that is $t_{k+q} < t+\tau < t_{k+q+1}$, then

$$\begin{aligned} \|\Psi_1(t+\tau) - \Psi_1(t)\| &\leq \left\| \sum_{0<t_k<t+\tau} \mathcal{S}_{\mu,\nu}(t+\tau-t_k)a_k - \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)a_k \right\| \\ &\leq \sum_{0<t_k<t} \|\mathcal{S}_{\mu,\nu}(t-t_k)\| \|a_{k+q} - a_k\| \\ &< \sum_{0<t_k<t} \frac{M}{1+|\omega|\left[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu\right]}\epsilon \\ &< \sum_{0<t_k<t} \frac{M}{|\omega|(t-t_k)^{\mu+1}}\epsilon \\ &\leq \frac{M\epsilon}{|\omega|}\left(\sum_{0<t-t_k\leq 1} \frac{1}{(t-t_k)^{\mu+1}} + \sum_{j=1}^{\infty} \sum_{j<t-t_k\leq j+1} \frac{1}{(t-t_k)^{\mu+1}} \right) \\ &\leq \frac{2MN\epsilon}{|\omega|}\left[m^{-(\mu+1)} + \sum_{j=1}^{\infty} \frac{1}{j^{\mu+1}} \right], \end{aligned}$$

where $m = \min\{t-t_k, 0 < t-t_k \leq 1\}$, N is a constant in the Lemma 2.19. Hence $\Psi_1(t) \in \mathcal{AP}(\mathbb{R},\mathbb{X})$.

Next, we show that $\Psi_2(t) \in \mathcal{PAP}_\rho(\mathbb{R}^+,\mathbb{X})$. By Lemma 2.18 there exists $g(k) = b_k$, $k \in \mathbb{Z}^+$, then

$$\begin{aligned} &\lim_{\gamma\to\infty} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \left\| \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)b_k \right\| \rho(t)dt \\ &\leq \lim_{\gamma\to\infty} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \sum_{0<t_k<t} \frac{M}{1+|\omega|[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu]} \|b_k\|\rho(t)dt \\ &\leq \lim_{\gamma\to\infty} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \sum_{0<t-t_k\leq 1} \frac{M}{1+|\omega|[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu]} \|g(t)\delta(t-k)\|\rho(t)dt \\ &\quad + \lim_{\gamma\to\infty} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \sum_{j=1}^{\infty} \sum_{j<t-t_k\leq j+1} \frac{M}{1+|\omega|[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu]} \|g(t)\delta(t-k)\|\rho(t)dt \\ &\leq \lim_{\gamma\to\infty} \frac{M}{|\omega|} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \sum_{0<t-t_k\leq 1} \frac{1}{(t-t_k)^{\mu+1}} \|g(t)\|\rho(t)dt \\ &\quad + \lim_{\gamma\to\infty} \frac{M}{|\omega|} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \sum_{j=1}^{\infty} \sum_{j<t-t_k\leq j+1} \frac{M}{(t-t_k)^{\mu+1}} \|g(t)\|\rho(t)dt \\ &\leq \lim_{\gamma\to\infty} \frac{M}{|\omega|} \frac{1}{w(\gamma,\rho)} \int_0^\gamma \frac{2N}{m^{\mu+1}} \|g(t)\|\rho(t)dt + \lim_{\gamma\to\infty} \frac{M}{|\omega|} \frac{1}{w(\gamma,\rho)} \int_0^\gamma K_0 \|g(t)\|\rho(t)dt = 0, \end{aligned}$$

ISSN: 2005-4297 IJCA

where $K_0 = \sum_{j=1}^{\infty} \sum_{j<t-t_k\leq j+1} \frac{1}{(t-t_k)^{\mu+1}} = \sum_{j=1}^{\infty} \frac{2N}{j^{\mu+1}}$. Hence $\Psi_2(t) \in \mathcal{PAP}_\rho(\mathbb{R}^+, \mathbb{X})$. □

Theorem 3.4. *Let $0 < \mu \leq \nu \leq 1$, $\beta > 0$ be given. Then under the assumptions $(H_1) - (H_4)$ the system $(1.1) - (1.3)$ admits a unique mild solution $y \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$ if*

$$\Theta := \left[\frac{ML_f|\omega|^{-1/(\mu+1)}\pi}{(\mu+1)\sin(\pi/(\mu+1))} + \frac{2MNL_I}{|\omega|}\left(m^{-(\mu+1)} + \sum_{j=1}^{\infty}\frac{1}{j^{\mu+1}}\right)\right] < 1. \tag{3.2}$$

Proof. Let $\mathcal{S}_{\mu,\nu}$ be the $(\mu,\nu)_\beta-$ regularized family generated by A. We define the operator $Q_{\mu,\nu}$ in the space $\mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$ by

$$(Q_{\mu,\nu}y)(t) = \sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)I_k(y(t_k)) + \int_0^t \mathcal{S}_{\mu,\nu}(t-s)f(s,y(s))ds.$$

By Theorem 2.20, we conclude that the function $s \to f(s,y(s))$ is in $\mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$, then by Lemma 3.2 we arrive at the conclusion that $\int_0^t \mathcal{S}_{\mu,\nu}(t-s)f(s,y(s))ds$ is in $\mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$. Similarly by Theorem 2.21 and Lemma 3.3 we come to the conclusion that $\sum_{0<t_k<t} \mathcal{S}_{\mu,\nu}(t-t_k)I_k(y(t_k))$ is in $\mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$.

Moreover, let $y, z \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$, we have

$$\begin{aligned}
&\|(Q_{\mu,\nu}y)(t) - (Q_{\mu,\nu}z)(t)\| \\
&\quad\leq ML_f\|y-z\| \int_0^t \frac{1}{1+|\omega|[(t-s)^{\mu+1} + \beta(t-s)^\nu)} ds \\
&\quad\quad + ML_I\|y-z\| \sum_{0<t_k<t} \frac{1}{1+|\omega|[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu)} \\
&\quad< ML_f\|y-z\| \int_0^\infty \frac{1}{1+|\omega|[s^{\mu+1}]} ds + ML_I\|y-z\| \sum_{0<t_k<t} \frac{1}{1+|\omega|[(t-t_k)^{\mu+1} + \beta(t-t_k)^\nu)} \\
&\quad\leq \left[\frac{ML_f|\omega|^{-1/(\mu+1)}\pi}{(\mu+1)\sin(\pi/(\mu+1))} + \frac{2MNL_I}{|\omega|}\left(m^{-(\mu+1)} + \sum_{j=1}^{\infty}\frac{1}{j^{\mu+1}}\right)\right]\|y-z\|.
\end{aligned}$$

Thus by Banach contraction principle, $Q_{\mu,\nu}$ has a unique fixed point in $\mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$ which is the mild solution of the system (1.1)-(1.3). □

4. Example

In this section, we consider the nonlinear two-term time-fractional impulsive diffusion wave equation with time operator in Caputo sense and nonlinear forcing term $f \in L^\infty_{loc}([0,T) \times \mathbb{R}), T > 0, u \in \mathbb{R}$ of the form

$$\begin{aligned}
a_1 D_t^{\alpha_1} z(t,u) + a_2 D_t^{\alpha_2} z(t,u) =& \frac{\partial^2}{\partial u^2} z(t,u) - b_0 z(t,u) \\
&+ D_t^{\alpha_1 - 1} f(t,u,z(t,u)), \quad t \in \mathbb{R}^+,\ t \neq t_k, u \in (0,\pi),
\end{aligned} \tag{4.1}$$

$$\Delta z(t_k,u) = z(t_k^+,u) - z(t_k^-,u) = \mathcal{G}_k(z(t_k,u)), \quad k \in \mathbb{Z}^+, u \in [0,\pi], \tag{4.2}$$

$$z(0,u) = f(u), \qquad z_t(0,u) = g(u), \quad f,g \in L^p(\mathbb{R}), 1 \leq p \leq \infty. \tag{4.3}$$

Recently, Mirjana and Rudolf [19] show the existence of sub viscosity and super viscosity solutions of the system (4.1)-(4.3) on the Lebesgue space $L^p(\mathbb{R}), 1 \leq p \leq \infty$ in case of $\mathcal{G}_k = 0$, $k \in \mathbb{Z}^+$. For convenience, we take $\mathbb{X} = L^2([0,\pi]), a_1 = 1, a_2 = \beta, \alpha_1 = \mu+1, \alpha_2 = \nu, b_0 > 0$ with $0 < \mu \leq \nu \leq 1$ and

ISSN: 2005-4297 IJCA

$f \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$. Let $y(t) = z(t, \cdot)$ and set $y(0) = y'(0) = 0$. Then the system (4.1)-(4.3) is in the abstract form of system (1.1)-(1.3).

$$D_t^{\mu+1} y(t) + \beta D_t^\nu y(t) = Ay(t) + D_t^\mu f(t, y(t)),$$
$$\Delta y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k \in \mathbb{Z}^+,$$
$$y(0) = y_0, y'(0) = y_1$$

where $Ay(t) := \frac{\partial^2}{\partial u^2} y(t) - b_0 y(t)$ with the domain $D(A) = \{u \in L^2[0, \pi] : u'' \in L^2[0, \pi], u(0) = u(\pi) = 0\}$. By Example 6.1 in [13], we known that $A = L - b_0$ is a sectorial operator with $\omega = -b_0 < 0$ of angle $\frac{\pi}{2}$(and hence of angle $\frac{\nu\pi}{2}$ for all $\nu \leq 1$) hence (H_2) holds. In addition, take $t_k = k + \frac{1}{4}|\sin 3k + \sin \sqrt{3}k|$, $I_k = \mathcal{G}_k$ and $\mathcal{G}_k \in \mathcal{WPAP}_\rho(\mathbb{Z}^+, \mathbb{X})$, $\rho \in W_T$, then (H_3) holds with $L_I = \sup_{k \in \mathbb{Z}^+} \|\mathcal{G}_k\|$. Note that $\{t_k^l\}$, $k, l \in \mathbb{Z}^+$ are equipotentially almost periodic such that $\kappa = \inf_{k \in \mathbb{Z}}(t_{k+1} - t_k) > 0$, thus (H_1) is satisfied, for more details see [17, 24, 27]. Now, we choose f defined by

$$f(t, y) = \frac{1}{16}(\sin 2t + \sin \sqrt{3}t + e^{-t} + g(t)) \sin y(t),$$

where $g(t) \in \mathcal{UPC}(\mathbb{R}^+, \mathbb{R}^+,)$ satisfies $|g(t)| \leq 1$ and $\lim_{\gamma \to \infty} \frac{1}{w(\gamma, e^{\eta t})} \int_0^\gamma |g(t)| e^{\eta t} dt = 0$. Then $f \in \mathcal{WPAP}_{e^{\eta t}}(\mathbb{R}^+, \mathbb{X})$, for $\eta > 0$ and (H_4) holds with $L_f = \frac{1}{4}$. Now, the following theorem is an immediate consequence of Theorem 3.4.

Theorem 4.1. *Let $0 < \mu \leq \nu \leq 1$, $\beta > 0$, be given. Then under the assumptions $(H_1) - (H_4)$ the system $(4.1) - (4.3)$ admits a unique mild solution $y \in \mathcal{WPAP}_\rho(\mathbb{R}^+, \mathbb{X})$, if*

$$\Theta := \left[\frac{M|\omega|^{-1/(\mu+1)}\pi}{4(\mu+1)\sin(\pi/(\mu+1))} + \frac{2MN \sup_{k \in \mathbb{Z}^+} \|\mathcal{G}_k\|}{|\omega|}\left(m^{-(\mu+1)} + \sum_{j=1}^{\infty} \frac{1}{j^{\mu+1}}\right)\right] < 1. \tag{4.4}$$

Acknowledgment

The work of the first author is supported by the "Ministry of Human Resource and Development, India under grant number:MHR-02-23-200-44".

Reference

[1] A.I. Alonso, J. Hong, R. Obaya; *Almost periodic type solutions of differential equations with piecewise constant argument via almost periodic type sequences.* Appl. Math. Lett. 13 (2000), 131–137.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander; *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, 96. Birkhäuser, Basel, (2001).

[3] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo; *Fractional Calculus Models and Numerical Methods, in: Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, (2012).

[4] B. Baeumer, S. Kurita, M. M. Meerschaert; *Inhomogeneous fractional diffusion equations.* Fract. Calc. Appl. Anal. 8 (2005), 371–386.

[5] Y. K. Chang, R. Ma, Z.H. Zhao; *Almost periodic solutions to a stochastic differential equation in Hilbert spaces*, Results in Math., 63(2013), 435–449.

[6] Y. K. Chang , R. Zhang, G. M. N'Guérékata; *Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations*, Comput. Math. Appl. 64(2012), 3160–3170.

[7] Y. K. Chang, G. M. N'Guérékata, Z. H. Zhao; *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients*, Nonlinear Anal. Theory Methods Appl. 72(2010), 430–438.

[8] A. M. Samoilenko, N. A. Perestyuk; *Impulsive Differential Equations*, vol. 14, World Scientific, Singapore, 1995.

[9] J. Dabas, A. Chauhan; *Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay.* Math. Comput. Modelling 57 (2013), 754–763.

[10] K. Diethelm; *The analysis of fractional differential equations.* Springer, New York (2010).

[11] M. M. El-Borai, A. Debbouche; *Almost periodic solutions of some nonlinear fractional differential equations*, Int. J. Contemp. Math. Sci. 4 (2009), 1373–1387.

[12] T. Guendouzi, L. Bousmaha; *Almost periodic solutions for impulsive fractional stochastic evolution equations*, Int. J. Anal. appl., 6(2014), 28–43.

[13] V. Keyantuo, C. Lizama, M. Warma; *Asymptotic behavior of fractional order semilinear evolution equations*, Diff. Integral Eq. 26(7/8)(2013), 757-780.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies 204, Elsevier Science B.V., Amsterdam, 2006.

[15] M., Lakshman, S. Abbas; *PC-almost automorphic solution of impulsive fractional differential equations*, Mediterr. J. Math. 12 (2015), 771–790.

[16] C. Lizama; *An operator theoretical approach to a class of fractional order differential equations.* Appl. Math. Lett. 24 (2011), 184–190.

[17] J. W. Liu, C. Y. Zhang, *Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations*, Adv. Diff. Eqn. 2013 (2013), 1–21.

[18] F. Mainardi; *The fundamental solutions for the fractional diffusion-wave equation.* Appl. Math. Lett. 9 (1996), 23–28.

[19] S. Mirjana, G. Rudolf; *Nonlinear two-term time-fractional diffusion-wave problem.* Nonlinear Anal. Real World Appl. 11 (2010), 3512–3523.

[20] K. S. Miller, B. Ross; *An introduction to the fractionalcal cululus and fractional differential equations*, Wiley, NewYork, (1993).

[21] E. A. Pardo, C. Lizama; *Weighted pseudo almost automorphic mild solutions for two-term fractional order differential equations* Appl. Math. Comp. 271 (2015), 154–167.

[22] E. A. Pardo, C. Lizama; *Pseudo asymptotic solutions of fractional order semilinear equations*, Banach J. Math. Anal. 7 (2013), 42–52.

[23] I. Podlubny; *Fractionl differential equations* Academic Press, New York (1999).

[24] A.M. Samoilenko, N.A. Perestyuk; *Impulsive defferential equations*, World Scientific, Singapore, (1995). Comput. Math. Appl. 64 (2012), 2100-2110.

[25] L. V. Trong; *Decay mild solutions for two-term time-fractional differential equations in Banach spaces.* J. Fixed Point Theory Appl. 18 (2016), 417–432.

[26] Z. Xia, D. Wang; *Piecewise weighted pseudo almost periodic solutions of impulsive integro-differential equations via fractional operators,* Elect. J. Diff. Equn. 185(2015) 1-18.

[27] Z. Xia; *Pseudo Almost Periodic Mild Solution of Nonautonomous Impulsive Integro-Differential Equations.* Mediterr. J. Math. 13 (2016), no. 3, 1065–1086.

[28] Z. Xia; *Pseudo asymptotically periodic solutions of two-term time-fractional differential equations with delay.* Kodai math. J. 38(2015), 310–332.

[29] Xia, Zhinan, Dingjiang Wang; *Pseudo- Almost Periodic Solution for Impulsive Hematopoiesis Model with Infinite Delays and Linear Harvesting Term*, International Journal of Biomathematics, 9 (2016), 1-17.