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SPANNING SET LABELING OF GRAPHS

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Abstract
In context of contribution to the topic here in this paper, we introduce a versatile concept of labeling. For a graph $G(V;\ E)$ where $|V\ | =\ n$, we constructed a spanning set of n on Galois field of two elements $GF(2)$ and with the help of the power set of this spanning set, we labeled all the vertices and edges of G. We explained significance as well as the limitations of this labeling. Applied it to various graphs and obtained several results and properties.
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1. Introduction

The graphs considered in this paper are only finite and simple. For the basic and useful concept of graph theory, its notation, and most popular definitions one can follow the book by D. West [6]. For an extensive survey on graph labeling, identification of major problems and results, see Gallian [3]. Mainly, the contribution of Cahit [1], Cichacz [2], Hovey [4], Pechenik [5] in the area of cordial labeling of graph established a milestone in the field of graph labeling.

Graph labeling can categories in three sections either vertex labeling or edge labeling or labeling of both vertices and edges. For all these labelings of a graph, it requires to obey certain conditions. Often been motivated by their significance in various fields of science and engineering, an enormous amount of literature has developed on several kinds of labeling of graphs. Set colorings of graphs were described by many authors, among them Hegde [10], Hopkroft and Krishnamurty [7], Balister et al. [9], and Acharya [8] presented many interesting and significant work. In these papers, they have taken a random set and with the help of its power set they obtained certain types of coloring of graphs.

In this paper, instead of random set we have chosen a spanning set $S = \{2^r \mid r = 0,\ 1,\ 2, \dots, n\}$ on $GF(2)$ where $n = |V|$ and introduce spanning set labeling of graphs: Let S be a nonempty spanning set corresponding to n, then power set of S contains $2^{|S|}$ elements (i.e., if $S = \{1,\ 2,\ 4\}$ then power set $P(S) = \{\phi\ ,\ \{1\},\ \{2\},\ \{4\},\ \{1,2\},\ \{1,3\}, \{2,4\},\ \{1,2,4\}\}$) from which we construct a set of labels $W(S) = P(S)/\ \{\phi\}$. For any two subsets A and B of W(S), $A \oplus\ B$ denote the symmetric difference of A, B and $A \oplus\ B$ is given by: $A \oplus\ B =\ (A \cup\ B) - (A \cap\ B)$.

For a given graph $G(V, E)$ and $|V| = n$ we obtain a nonempty spanning set S of n on $GF(2)$, we define a function f on vertex set V of G with an assignment of subsets of S to the vertices of G, and with the help of f we define $f^{\oplus}$ on the edge set E of G i.e., $f^{\oplus} :\ V \times\ V \rightarrow\ E\ \ such\ that\ f^{\oplus}(uv)\ = f(u) \oplus f(v)$.

Let $f(G) = \{f(u): u \in\ V\}$ and $f^{\oplus}(G) = \{f^{\oplus}(e) :\ e \in\ G\ \}$. We say f a spanning set labeling of G if both f and $f^{\oplus}$ are injective functions. A graph is called spanning set label if it admits a spanning set labeled. A spanning set labeling f of G is called a strong spanning set labeling if f and $f^{\oplus}$ are disjoint subsets of S.

A spanning set labeling is called proper if $f^{\oplus}\ = W(S)$. The spanning set chromatic number $\sigma(G)$ is the least cardinality of a set S with respect to which G has a set labeling. Further, if $f: V \rightarrow\ S$ is a spanning set labeling of G with $|S| = \sigma(G)$ we call f an optimal spanning set labeling of G.

Proposition 1: *For any graph G,* $1 \leq\ \sigma(G) \leq \lfloor log_2\, n\rfloor + 1$. *Where* $\lfloor\ x\rfloor$ *is the floor function of a real number x and the bounds are best possible.*

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This time, various types of labeling present in the literature for different types of graphs other than complete graphs. This article provides a wide range of applications for the proper labeling of complete graphs. The concept proposed here is suitable for all types of graphs and it provides extremal labeling of any graphs. Some useful theorems and results are given below:

Definition 1:

For any graph G the degree sum of any vertex for spanning set labeling is the union of all edges labels incident to that vertex along with that vertex label. That is, if A_i is vertex label set of vertex V_i and B_1, B_2, B_3 are edge label sets corresponding to edges e_1, e_2, e_3 incident to vertex V_i then degree sum of vertex V_i is set $A_i \cup B_1 \cup B_2 \cup B_3$.

Definition 2:

A spanning set labeling graph is said to be ***antimagic*** if the degree sum of every vertex is distinct.

Definition 3:

If the degree sum of all vertices remains the same then the spanning set labeling of such a graph is called ***magic spanning set labeling*** of graphs.

Theorem 1:

Every path graph P_n admits spanning set labeling with extrimal bound $\sigma(G) \leq \lfloor log_2 n \rfloor$.

Proof: Let P_n is a complete graph with n vertices. So, there exist a spanning set $S = \{2^0, 2^1, 2^2, \dots, 2^r\}$ where $r = 0, 1, 2, \dots, \lfloor log_2 n \rfloor$. Thus, the cardinality of $|S| = \lfloor log_2 n \rfloor + 1$. Since, $m^{\{log_m n\}} = n\$$ this implies $2^{\{|S|\}} \geq n$ thus, for each vertex we easily get a different label set X i.e., for any $u \neq v \in V$, $\exists X \neq Y \in W(S)$ such that $f(u) \neq f(v)$. Hence, $V(G)$ admits spanning set labeling.

Now, if we label all the vertices of G with the distinct ordered power set of S then the symmetric difference between any two vertices is also the member of the power set of S. As in a path graph P_n there must be $(n-1)$ edges. Since for all n, $|W(S)| > (n-1)$ thus, G admits spanning set labeling for all n.

Example 1: Path graph P_5 admits spanning set labeling.

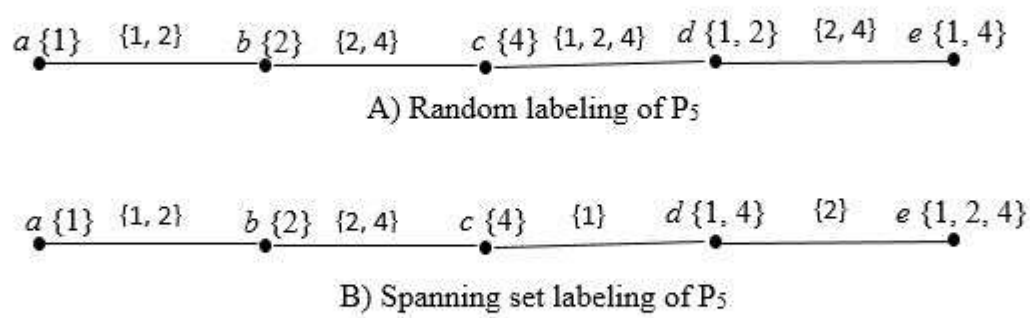

Figure 1: Labeling of P_5

In this example we can observe that graph A of figure 1 which is a random set labeling of path graph not satisfy the condition of labeling. Whereas graph B of this figure 1 fulfills the condition of set labeling. Thus, we developed an algorithm that ensures that any path graph holds spanning set labeling.

Algorithm for set labeling of any path graph:

Let we obtained $S = \{2^0, 2^1, 2^2, 2^3, \dots, 2^m\}$ so that $|S| = (m+1)$. Firstly we fixed the source vertex and assign it label $\{2^0\}$ then we label next successive vertices by the set $\{2^0, 2^1\}$, $\{2^0, 2^2\}$, ..., $\{2^0, 2^m\}$. Now we label $(m+2)^{th}$ vertex by $\{2^1\}$ and its successive vertices by the set $\{2^1, 2^2\}$, $\{2^2, 2^3\}$, $\{2^3,$

$2^4\}$, … , $\{2^{(m-2)},\ 2^{(m-1)}\}$. Similarly, we label $(2m+3)^{rd}$ vertex by $\{2^2\}$ and next $(m-1)$ successive vertex by the sets with cadinality three hence by $\{2^0,\ 2^3,\ 2^4\}$, $\{2^1,\ 2^4,\ 2^5\}$, $\{2^2,\ 2^5,\ 2^6\}\}$,..., $\{2^{(m-4)}$, $2^{(m-3)},\ 2^{(m-2)}\}$. This process continues till m^{th} last vertices. The last $(m-1)$ vertices can be labeled by the set S_1 with cardinality m then by set S_2 of cardinality one and next by the set of cardinality $(m-1)$ and its adjacent of cardinality two and so on. Following this pattern of labeling, we can obtain the set labeling of any path graph.

Theorem 2:

Every complete graph K_n admits spanning set labeling.

Proof: Let K_n is a complete graph with n vertices. So, there exist a spanning set $S=\{2^0,\ 2^1,\ 2^2,\ldots,\ 2^r\}$ where $r=0,\ 1,\ 2,\ldots,\ \lfloor log_2 n\rfloor$. Thus, the cardinality of $|S|=\lfloor log_2 n\rfloor+1$. Since, $m^{\{log_m n\}}=n$ this implies $2^{|S|}\geq n$ thus, for each vertex we easily get a different label set X i.e., for any $u\neq v\in V$, $\exists\ X\neq Y\in W(S)$ such that $f(u)\neq f(v)$. Hence, $V(G)$ admits spanning set labeling.

Now, if we label all the vertices of G with distinct ordered power set of S then the symmetric difference between any two vertices is also the member of the power set of S. As in a complete graph K_n there must be $\frac{n(n-1)}{2}$ edges. Since for $n\leq\ 4,\ |W(S)|>\frac{n(n-1)}{2}$ thus, G admits spanning set labeling for $n\leq 4$. For $n>4$ there may exists some edges with the same label but that always admits with any two non-adjacent edges because we already labeled all the adjacent vertices with distinct set.

Example 2: Complete graph K_5 admits spanning set labeling.

In this example we applied algorithm 1 to find the spanning set labeling of graph and we observed that this satisfies all concepts of graph labeling.

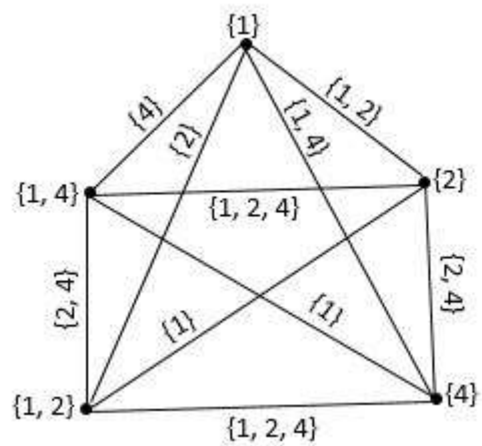

Figure 2: Labeling of K_5

Definition 4:

A graph is said to be ***strongly*** spanning set labeling if and only if all the vertices and edges can receive a distinct label set.

Corollary 1: *Every complete graph K_n for $n\leq 4$ is strongly spanning set labeled.*

Proof: We observed that for K_4 the spanning set has cardinality three. Thus, $|W(S)|=7$ so we have four independent choices for labeling. So it is obvious to assign a distinct label to all vertices and edges.

Corollary 2: *Every cycle graph C_n admits strongly spanning set labeling.*

Proof is obvious and followed from theorem 1 and 2.

Example 3: Cycle graph C_5 admits strongly spanning set labeling.

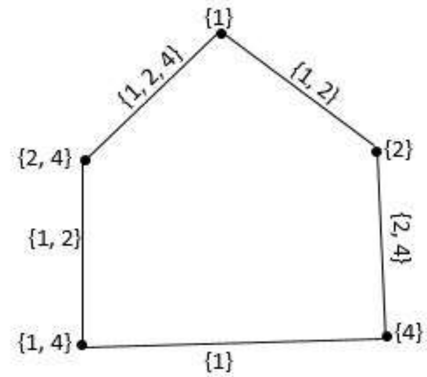

Figure 3: Labeling of C_5

In this example we have $S = \{1, 2, 4\}$ thus, we get \$$W(S) = \{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}$. Here we followed algorithm 1 and obtained this labeling.

Conclusion:

In this paper, we obtained a new concept of spanning set labeling of complete graphs, path graphs, and cycle graphs. This approach of labeling help to solve the assignment problems, problems related to game theory and chance measure of uncertain events. In the future, we like to extend this concept of labeling to some special graphs and obtain magic, antimagic properties of this labeling.

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