Certain inequalities for the ℓ −Hypergeometric Function

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Abstract

In this paper, we investigate certain inequalities for the ℓ-Hyper-geometric function and ℓ-Hypergeometric exponential function and mention a few consequences of our main results. A nonlinear differential equation involving the ℓ-Hyper-geometric function is also investigated.

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1 Introduction

Let $\mathcal{A}(\mathbb{D}_1(0))$ denote the class of analytic functions in the open unit disk $\mathbb{D}_1(0) = \{z \in \mathbb{C}: |z| < 1\}$. Let $\mathcal{C}$ be the class of all functions $f \in \mathcal{A}(\mathbb{D}_1(0))$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ and have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_1(0). \tag{1}$$

Two functions $f, g \in \mathcal{A}(\mathbb{D}_1(0))$ we say that f is subordinated to g in $\mathbb{D}_1(0)$ and express symbolically $f(z) \prec g(z)$, if there exists a function $\omega \in \mathcal{A}(\mathbb{D}_1(0))$ with $|\omega(z)| < |z|$, $z \in \mathbb{D}_1(0)$ such that $f(z) = g(\omega(z))$ in $\mathbb{D}_1(0)$. Furthermore, if function f is univalent in $\mathbb{D}_1(0)$, then g is subordinate to f provided $g(0) = f(0)$ and $g(\mathbb{D}_1(0)) \subset f(\mathbb{D}_1(0))$. By $\mathcal{S}$ we denote the class of all functions in $\mathcal{C}$ which are univalent in $\mathbb{D}_1(0)$. Let $\mathcal{S}^*(\varepsilon)$, $\mathcal{C}(\varepsilon)()$, $\mathcal{K}(\varepsilon)()$, $\tilde{\mathcal{S}}^*(\varepsilon)$ and $\tilde{\mathcal{C}}(\varepsilon)$ denote the classes of starlike, convex, close-to-convex, strongly starlike and strongly convex functions of order ε, respectively, and are defined as

$$\mathcal{S}^*(\varepsilon) = \left\{f \in \mathcal{C}: \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \varepsilon, z \in \mathbb{D}_1(0), 0 \le \varepsilon < 1\right\},$$

$$\mathcal{C}(\varepsilon) = \left\{f \in \mathcal{C}: \mathcal{R}e\left(\frac{(zf'(z))'}{f'(z)}\right) > \varepsilon, z \in \mathbb{D}_1(0), 0 \le \varepsilon < 1\right\},$$

$$\mathcal{K}(\varepsilon) = \left\{f \in \mathcal{C}: \mathcal{R}e\left(\frac{zf'(z)}{g(z)}\right) > \varepsilon, z \in \mathbb{D}_1(0), 0 \le \varepsilon < 1, g \in \mathcal{S}^*(0) \equiv \mathcal{S}^*\right\},$$

$$\tilde{\mathcal{S}}^*(\varepsilon) = \left\{f \in \mathcal{C}: \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\varepsilon\pi}{2}, z \in \mathbb{D}_1(0), 0 \le \varepsilon < 1\right\},$$

and

$$\tilde{\mathcal{C}}(\varepsilon) = \left\{f \in \mathcal{C}: \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\varepsilon\pi}{2}, z \in \mathbb{D}_1(0), 0 \le \varepsilon < 1\right\}.$$

For more details regarding these classes see [5, 7].

Recently, many researchers studied classes of analytic functions involving special functions $F \subset \mathcal{A}$, to find sufficient conditions such that the members of F have geometric properties like univalency,

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starlikeness or convexity in $\mathbb{D}_1(0)$. In this line many works are available in the literature, for generalized hypergeometric functions [15, 19, 26], Bessel functions [1, 2, 23, 27], Wright functions [17, 18, 24, 13]. In this paper, we study geometric properties of the ℓ-Hypergeometric function(ℓ-H function). For $z \in \mathbb{C}$, the ℓ-H function is defined as

$$H\begin{bmatrix} a; & z \\ b; (c:\ell) & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{2}$$

where $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$, $a, \ell \in \mathbb{C}$ with $\mathcal{Re}(\ell) \geq 0$ and $b, c \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$. If we put $\ell = 0$ in (2), then ℓ-H function turns to well known confluent hypergeometric function,

$$H\begin{bmatrix} a; & z \\ b; (c:0) & \end{bmatrix} = {}_1F_1\begin{bmatrix} a; & z \\ b; & \end{bmatrix}. \tag{3}$$

The ℓ-H function (2) recently studied by M. H. Chudasama and B. I. Dave [4].

Observe that ℓ-H function (2) does not belong to the family $\mathcal{C}$. Thus, it is natural to consider the following normalization of ℓ-H function:

$$\begin{aligned} \mathcal{H}(a; b; (c, \ell); z) &= zH\begin{bmatrix} a; & z \\ b; (c:\ell) & \end{bmatrix} \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{z^n}{(n-1)!}. \end{aligned} \tag{4}$$

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which are useful in our main results. In section 3, we derive sufficient conditions for the univalency of ℓ-H function.

2 Preliminaries

In this section we present few definitions and lemmas which are useful in the sequel. **Lemma 2.1. [4]** If $\mathcal{Re}(\ell) \geq 0$ and $\mathcal{Re}\left(c\,\ell - \frac{\ell}{2} + 1\right) > 0$, then the ℓ-H function is an entire function of z.

Definition 2.1. [4] Let $f(z) = \sum_{n=1}^{\infty} a_n z^n, z \in \mathbb{C}\backslash\{0\}, k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{C}$. Define

$${}_k\Delta_p^{\mathfrak{D}} f(z) = \begin{cases} \sum_{n=1}^{\infty} a_n (p)_{n-1}^k (\mathfrak{D} + p - 1)^{kn} z^n, & \text{if } k \in \mathbb{N}, \\ f(z), & \text{if } k = 0, \end{cases}$$

where $\mathfrak{D}$ is the Euler differential operator given by $\mathfrak{D} = z\frac{d}{dz}$.

From the above definition it can be seen that the ℓ-H function (2) satisfies the differential equation

$$({}_\ell\Delta_c^{\mathfrak{D}})(\mathfrak{D} + b - 1)\mathfrak{D}w - z(\mathfrak{D} + a)w = 0, \tag{5}$$

for $\ell \in \mathbb{N} \cup \{0\}, a, z \in \mathbb{C}$ and $b, c \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$. It was established in [4].

Definition 2.2. [4] The ℓ-H exponential function is defined as

$$e_H^{\ell}(z) = \mathcal{H}(-; -; (1, \ell); z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}}, \tag{6}$$

for all $z \in \mathbb{C}$ and $\mathcal{Re}(\ell) \geq 0$.

In this paper, we investigate certain inequalities for the ℓ-H function $\mathcal{H}(a; b; (c, \ell); z)$. We, also study an initial value problem involving the ℓ-H function.

Lemma 2.2 [6] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$. If the sequence $\{a_n\}$ is convex decreasing, i.e., $a_{n+1} - a_n \leq a_{n+2} - a_{n+1} \leq 0$ for all $n \in \mathbb{N}\backslash\{1\}$, then

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$$\mathcal{R}e(\sum_{n=1}^{\infty} a_n z^{n-1}) > \frac{1}{2}, z \in \mathbb{D}_1(0).$$

Definition 2.3. [3] The convex hull of $\mathcal{C}$, denoted by $\overline{Con}\mathcal{C}$, is the set of all convex combinations of functions belonging to $\mathcal{C}$ and is given by

$$\overline{Con}\mathcal{C} = \left\{ f \in \mathcal{C}: \mathcal{R}e\left(\frac{f(z)}{z}\right) > \frac{1}{2}, z \in \mathbb{D}_1(0) \right\}. \tag{7}$$

Definition 2.4. [29] The sequence $\{b_n\}$ of complex numbers is said to be a subordering sequence for the class $\mathcal{X} \subset \mathcal{C}$, if

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec \sum_{n=1}^{\infty} a_n z^n, z \in \mathbb{D}_1(0)$$

for all $\sum_{n=1}^{\infty} a_n z^n \in \mathcal{X}$.

Lemma 2.3. [22] The function of the form (1) is in the set $\overline{Con}\mathcal{C}$ if and only if $a_2, a_3, a_4, \cdots$ is a subordering factor sequence for the class $\mathcal{C}$.

3 Main Results

Theorem 3.1. If $a, b, c, \ell \in \mathbb{R}$ with $\ell \geq 1$ and $c \geq 1$. Then

$$|\mathcal{H}(a; b; (c, \ell); z)| \leq r \quad {}_1F_1\begin{bmatrix} a; & r \\ b; & \end{bmatrix},$$

where ${}_1F_1\begin{bmatrix} a; & r \\ b; & \end{bmatrix}$ is the confluent hypergeometric function given in (3) and $|z| = r < 1$.

Proof. Since $c \geq 1$ and $\ell \geq 1$, it follows that $\frac{1}{(c)_n^{\ell n}} \leq 1$. Thus,

$$\begin{aligned} |\mathcal{H}(a; b; (c, \ell); z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{|z|^n}{(n-1)!} \\ &\leq |z| + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}} \frac{|z|^n}{(n-1)!} \\ &= r \quad {}_1F_1\begin{bmatrix} a; & r \\ b; & \end{bmatrix}. \end{aligned}$$

This completes the proof.

Corollary 3.1. If $a, b, c, \ell \in \mathbb{R}$ with $\ell \geq 1$ and $c \geq 1$. Then

$$\left| H\begin{bmatrix} a; & z \\ b; (c:\ell) & \end{bmatrix} \right| \leq {}_1F_1\begin{bmatrix} a; & r \\ b; & \end{bmatrix}, |z| = r < 1.$$

Proof. From (3) and Theorem 3.1, we get the result.

Theorem 3.2. If $a, b, c, \ell \in \mathbb{R}$ with $0 < a \leq b, \ell \geq 1$ and $c \geq 1$. Then

$$|\mathcal{H}(a; b; (c, \ell); z)| \leq r e_H^{\ell}(r)$$

where $e_H^{\ell}(r)$ is the ℓ-H exponential function given in (6) and $|z| = r < 1$.

Proof. Since $c \geq 1$ and $\ell \geq 1$, it follows that $\frac{1}{(c)_n^{\ell n}} \leq \frac{1}{(n!)^{\ell n}}$. Thus,

$$\begin{aligned} |\mathcal{H}(a; b; (c, \ell); z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{|z|^n}{(n-1)!} \\ &\leq |z| \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^{\ell n}} \frac{|z|^n}{(n)!} \right] \\ &= r e_H^{\ell}(r). \end{aligned}$$

This completes the proof.

Corollary 3.2. If $a, b, c, \ell \in \mathbb{R}$ with $\ell \geq 1$ and $c \geq 1$. Then

$$\left| H\begin{bmatrix} a; & z \\ b; (c:\ell) & \end{bmatrix} \right| \leq e_H^{\ell}(r), |z| = r < 1.$$

Proof. From (2), (6) and Theorem 3.2, we get the result.

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Theorem 3.3. If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b, \ell \geq 1$ and $c \geq 2^{1/\ell}$. Then

$$\mathcal{R}e\left(\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right) > \frac{1}{2}, z \in \mathbb{D}_1(0).$$

Proof. Since $b > a, c \geq 2^{1/\ell}$, it follows that $bc^\ell \geq 2a$. So,

$$\begin{aligned} a_3 - 2a_2 + a_1 &= \frac{(a)_2}{2(b)_2(c)_2^{2\ell}} - \frac{2a}{bc^\ell} + 1 \\ &= \frac{(a)_2}{2(b)_2(c)_2^{2\ell}} + \frac{bc^\ell - 2a}{bc^\ell} \geq 0. \end{aligned} \tag{8}$$

Since $b > a$ and $c \geq 2^{1/\ell} \geq 0$, it follows that

$$(c + n - 1)^n \geq \frac{1}{(c)_{n-1}} = \frac{\Gamma(c)}{\Gamma(c+n-1)}, n \in \mathbb{N},$$

and

$$n(c)_{n-1}^\ell \geq \frac{2(a+n-1)}{(b+n-1)(c)_n^{\ell n}}.$$

Therefore,

$$\begin{aligned} a_{n+2} & -2a_{n+1} + a_n \\ &= \frac{(a)_{n+1}}{(b)_{n+1}(c)_{n+1}^{\ell(n+1)}(n+1)!} - \frac{2(a)_n}{(b)_n(c)_n^{\ell n} n!} + \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}(n-1)!} \\ &= \frac{(a)_{n+1}}{(b)_{n+1}(c)_{n+1}^{\ell(n+1)}(n+1)!} \\ & \quad 2cm + \frac{(a)_{n-1}}{(b)_n(c)_{n-1}^{\ell(n-1)} n!}\left[n(c)_{n-1}^\ell - \frac{2(a+n-1)}{(b+n-1)(c)_n^{\ell n}}\right] \\ &\geq 0, \end{aligned}$$

for $n \in \mathbb{N}\backslash\{1\}$. From (8) and the above identity, it follows that the sequence $\{a_n\}$ is a convex decreasing for $n \in \mathbb{N}$. Hence, from Lemma 2, we get

$$\mathcal{R}e\left(\sum_{n=1}^\infty a_n z^{n-1}\right) > \frac{1}{2}, z \in \mathbb{D}_1(0).$$

That is

$$\mathcal{R}e\left(\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right) > \frac{1}{2}, z \in \mathbb{D}_1(0).$$

Theorem 3.4. If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b, \ell \geq 1$ and $c \geq 2^{2/\ell}$. Then

$$\mathcal{R}e(\mathcal{H}'(a;b;(c,\ell);z)) > \frac{1}{2}, z \in \mathbb{D}_1(0).$$

Proof. The proof is similar to the proof of the Theorem 3.3. Hence, we omit details.

Corollary 3.3. If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b, \ell \geq 1$ and $c \geq 2^{1/\ell}$. Then the sequence

$$\left\{\frac{(a)_n}{(b)_n(c)_n^{\ell n} n!}\right\}_{n=1}^\infty \tag{9}$$

is a subordinating sequence for the class $\mathcal{C}$.

Proof. From Theorem 3.1 and (7), we have $\mathcal{H}(a;b;(c,\ell);z) \in \overline{Con}\mathcal{C}$. Hence, from Lemma 2.3, the sequence (9) is a subordinating sequence for the class $\mathcal{C}$.

Corollary 3.4. If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b, \ell \geq 1$ and $c \geq 2^{2/\ell}$. Then the sequence

$$\left\{\frac{(n+1)(a)_n}{(b)_n(c)_n^{\ell n} n!}\right\}_{n=1}^\infty \tag{10}$$

is a subordinating sequence for the class $\mathcal{C}$.

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Proof. From Theorem 3.1 and (7), we have $z\mathcal{H}(a;b;(c,\ell);z) \in \overline{Con}\mathcal{C}$. Hence, from Lemma 2.3, the sequence (9) is a subordinating sequence for the class $\mathcal{C}$.

4 Nonlinear differential equation

In this section, we study a nonlinear differential equation which involves the ℓ-Hypergeometric function. In this purpose, we required the following lemmas.

Lemma 4.1. [14] Let $\Omega \subset \mathbb{C}$. Suppose that the function $\psi(z): \mathbb{C} \times \mathbb{C} \times \mathbb{D}_1(0) \to \mathbb{C}$ satisfies the condition $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq M(M-|a|)/(M+|a|), \theta \in \mathbb{R}$ and $z \in \mathbb{D}_1(0)$. Let $q(z)$ be an analytic function of the form

$$q(z) = a + a_1 z + a_2 z^2 + \cdots, z \in \mathbb{D}_1(0) \tag{11}$$

such that $\psi(zp(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{D}_1(0)$. Then $|q(z)| < M$, where $0 \leq |a| < M \leq 1$.

Lemma 4.2. If $f \in \mathcal{C}$ and $|f''(z)| \leq 1, z \in \mathbb{D}_1(0)$, then f is starlike in $\mathbb{D}_1(0)$.

Theorem 4.1. Let $a, b, c, \ell \in \mathbb{R}$ and suppose $\mathcal{H}(a;b;(c,\ell);z)$ satisfy the inequality

$$|z\mathcal{H}(a;b;(c,\ell);z)| < \frac{M(M-|a|)}{(M+1)(M+|a|)}, 0 \leq |a| < M \leq 1, z \in \mathbb{D}_1(0).$$

Let u be the solution of the initial value problem

$$u^{(n+1)}(z) + \mathcal{H}(a;b;(c,\ell);z)u^{(n)}(z) = \mathcal{H}(a;b;(c,\ell);z), z \in \mathbb{D}_1(0),$$
$$u(0) = 0, u'(0) = 1, u^{(k)}(0) = 0, k = 2,3,\cdots, n-1, u^{(n)}(0) = a.$$

Then $|u^{(n)}(z)| < M, z \in \mathbb{D}_1(0)$.

Proof. Let the function $q(z)$ be defined by $q(z) = u^{(n)}(z), z \in \mathbb{D}_1(0)$. Note that $q(z)$ has the form (11), and then it follows that

$$\frac{zq\prime(z)}{1+q(z)} = \frac{zu^{(n+1)}(z)}{1+u^{(n)}} = z\mathcal{H}(a;b;(c,\ell);z), z \in \mathbb{D}_1(0), u^{(n)}(z) \neq -1.$$

Define $\psi(l,m;z): \mathbb{C} \times \mathbb{C} \times \mathbb{D}_1(0) \to \mathbb{C}$ by

$$\psi(l,m;z) = \frac{m}{1+l}, l \neq -1$$

and

$$\Omega = \left\{w \in \mathbb{C}: |w| < \frac{M(M-|a|)}{(M+1)(M+|a|)}, 0 \leq |a| < M \leq 1\right\}.$$

Then,

$$\psi(zp(z), zp'(z); z) = \frac{zq\prime(z)}{1+q(z)} = \frac{zu^{(n+1)}(z)}{1+u^{(n)}} \in \Omega, z \in \mathbb{D}_1(0).$$

Now, for any $\theta \in \mathbb{R}, K \geq \frac{M(M-|a|)}{M+|a|}$ and $z \in \mathbb{D}_1(0)$, we have

$$\psi(Me^{i\theta}, Ke^{i\theta}; z) = \frac{Ke^{i\theta}}{1+Me^{i\theta}} \geq \frac{M(M-|a|)}{M+|a|}.$$

Thus,

$$\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega.$$

Hence from Lemma 4.1, $|u^{(n)}(z)| < M, 0 \leq |a| < M \leq 1, z \in \mathbb{D}_1(0)$.

Let $a, b, c, \ell \in \mathbb{R}$ and suppose $\mathcal{H}(a;b;(c,\ell);z)$ satisfy the inequality

$$|z\mathcal{H}(a;b;(c,\ell);z)| < \frac{M(M-|a|)}{(M+1)(M+|a|)}, 0 \leq |a| < M \leq 1, z \in \mathbb{D}_1(0).$$

Let u be the solution of the initial value problem

$$u'''(z) + \mathcal{H}(a;b;(c,\ell);z)u''(z) = \mathcal{H}(a;b;(c,\ell);z), z \in \mathbb{D}_1(0),$$
$$u(0) = 0, u'(0) = 1, u''(0) = a.$$

Then $|u^{(n)}(z)| < M, z \in \mathbb{D}_1(0)$.

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Proof. Taking $n = 2$ in the above Theorem 4.1, we get desired result.

Theorem 4.2. Let $a, b, c, \ell \in \mathbb{R}$ and suppose $\mathcal{H}(a; b; (c, \ell); z)$ satisfy the inequality

$$|z\mathcal{H}(a; b; (c, \ell); z)| < \frac{1-|a|}{2(1+|a|)}, 0 \leq |a| < 1, z \in \mathbb{D}_1(0).$$

Let u be the solution of the initial value problem

$$u'''(z) + \mathcal{H}(a; b; (c, \ell); z)u''(z) = \mathcal{H}(a; b; (c, \ell); z), z \in \mathbb{D}_1(0),$$
$$u(0) = 0, u'(0) = 1, u''(0) = a.$$

Then ψ is starlike in $z \in \mathbb{D}_1(0)$.

Proof. Taking $M = 1$ in the above Corollary 4.1 and using Lemma 4.2. We get desired result.

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