Fixed point theorems of Rus-Reich- Ciri´c type contraction and Hardy-Rogers type contraction on G-metric spaces

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Abstract: *In this manuscript, we present the notion of generalized interpolative Rus-Reich- Ciri´c type contraction and the notion of interpolative G-Hardy-Rogers type contraction and give some fixed point theorems of generalized Rus-Reich- Ciri´c type contraction and interpolative G-Hardy-Rogers type contraction.*

Introduction:

Banach [21] gave the base of metric fixed point theory by proposing his fixed point result in 1992. Based on this, Kannen [1] gave a new fixed point result in 1968. He considered the following contraction type

$$d(Tx,\ Ty) \le \lambda[d(x,\ Tx) + d(y,\ Ty)] \quad \text{For all } x, y \in X\ , \tag{1}$$

where $\lambda \in [0, \frac{1}{2})$. Very recently, in [2] the acclaimed theorem of Kannan was revisited by using the interpolation theory. For a metric space (X, d), the self-mapping $T : X \to X$ is said to be an interpolative Kannan type contraction, if there are constants $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \lambda[d(x,Tx)]^{\alpha}.[d(y,Ty)]^{1-\alpha}, \tag{2}$$

for all $x, y \in X \setminus Fix(T)$ *where* $Fix(T) = \{u \in X,\ Tu = u\}$ *with* $x \ne Tx$.

Erdal Karapinar et al. [19]. gave the notion of interpolative Reich-Rus- Ciri´c type contractions. They proposed the following contraction type in the framework of partial metric space (X, p), a mapping $T : X \to X$ is called an interpolative Reich-Rus- Ciri´c type contractions

if there are constants $\lambda \in [0,1)$ such that

$$\text{p(Tx, Ty)} \le \lambda\ [\text{p(x, y)}]^{\beta}[\text{p(x, Tx)}]^{\alpha}[\text{p(y, Ty)}]^{1-\alpha-\beta} \tag{3}$$

for all x, y $\in$ X \ Fix(T).

Results:

2. Generalized interpolative Rus-Reich- Ciri´c type contraction:

Definition 2.1: In the framework of a G-metric space (X, G), a mapping

$T : X \to X$ is called an Generalized interpolative Rus-Reich-*Ciri´c* type contraction, if there are constants $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ such that

$$G(Tx,Ty,Tz) \le \lambda\ [G(x,y,z)]^{\beta}.[G(x,Tx,Tx)]^{\alpha}.[G(y,Ty,Ty)]^{\gamma}.[G(z,Tz,Tz)]^{1-\alpha-\beta-\gamma} \qquad (2.1)$$

for all $x,y,z \in X \setminus Fix(T)$.

Theorem 2.1:

In the frame work of a G-metric space (X, G), if $T: X \to X$ *is* an Generalized interpolative Rus-Reich-*Ciri´c* type Contraction, then T has a fixed point in X .

Proof: Take an arbitary point $x_n = T^n(x_0) = Tx_{n-1}$ for each positive integer n. If there exists n_0 such that $x_{n_0} = x_{n_0} + 1$, then x_{n_0} is a fixed point of T. The proof is completed.

Hence forwards, assume that

$x_n \neq x_{n+1}$ for each $n \ge 0$. By substituting the values $x = x_n$, $y = x_{n-1}$ and $z = x_{n-1}$ on (3.1), we find that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\le \lambda[G(x_{n-1}, x_n, x_n)]^{\beta}.[G(x_{n-1}, Tx_{n-1}, Tx_{n-1})]^{\alpha}.[G(x_n, Tx_n, Tx_n)]^{\gamma}.[G(x_n, Tx_n, Tx_n)]^{1-\alpha-\beta-\gamma} \\ &= \lambda[G(x_{n-1}, x_n, x_n)]^{\beta}.[G(x_{n-1}, x_n, x_n)]^{\alpha}.[G(x_n, x_{n+1}, x_{n+1})]^{\gamma}.[G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha-\beta-\gamma} \\ &= \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha+\beta}.[G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha-\beta}. \end{aligned}$$

By a calculation, we derive

$$[G(x_{n-1}, x_n, x_n)]^{\alpha+\beta} \le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha+\beta} \qquad (2.3)$$

from the inequality (2.2). We conclude that $\{G(x_{n-1},x_n,x_n)\}$ is a non-increasing sequence with non-increasing terms. Thus, there is a non-negative constant 't' such that $\lim_{n\to\infty} G(x_{n-1}, x_n, x_n) = l$.

Note that $l \ge 0$. Indeed, from (2.3), we deduce that

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le \lambda G(x_{n-1}, x_n, x_n) \le \lambda^n G(x_0, x_1, x_1) \qquad (2.4)$$

Regarding $\lambda < 1$, and by taking $n \to \infty$ in the inequality (2.4), we deduce that l=0.

For what follows, we shall prove that $\{x_n\}$ is a fundamental (G-Cauchy) sequence by employing standard tools. More precisely, starting with the rectangular inequality, we shall get the following estimation

$$\begin{aligned} G(x_n,x_{n+r},x_{n+r}) &\le G(x_n,x_{n+1},x_{n+1}) + \ldots\ldots\ldots\ldots + G(x_{n+r-1},x_{n+r},x_{n+r}) \\ &\le \lambda^n G(x_0,x_1,x_1) + \ldots\ldots\ldots + \lambda^{n+r-1} G(x_0,x_1,x_1) \qquad (2.5) \\ &\le \frac{\lambda^n}{1-\lambda} G(x_0,x_1,x_1) \end{aligned}$$

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letting $n \to \infty$ in the inequality (2.5), we ascertain that $\{x_n\}$ is a G-Cauchy sequence.
Hence, $\lim_{n,m\to\infty} G(x_n,x_m,x_m)=0$, that is $\{x_n\}$ is a G-Cauchy sequence in the complex G-metric space(X,G) and so there exists $x \in X$ such that $\lim_{n\to\infty} G(x_n,x,x)=0$.
Suppose that $x \neq Tx$.Since $x_n \neq Tx_n$ for each $n \geq 0$, by letting $x=x_n$, $y = x$ *and* $z=x$ in (2.1), we have

$$
\begin{aligned}
G(x_{n+1},Tx,Tx)&=G(Tx_n,Tx,Tx)\\
&\leq [G(x_n,x,x)]^{\beta}[G(x_n,Tx,Tx_n)]^{\alpha}[G(x,Tx,Tx)]^{\gamma}[G(x,Tx,Tx)]^{1-\alpha-\beta-\gamma}\\
&\leq \lambda[G(x_n,x,x)]^{\beta}[G(x_n,x_{n+1},x_{n+1})]^{\alpha}[G(x,Tx,Tx)]^{1-\alpha-\beta} \qquad (2.6)
\end{aligned}
$$

letting $n \to \infty$ in the inequality (2.6), we get that $G(x,Tx,Tx)=0$, which is a contradiction.

Therefore our assumption is wrong i.e $x=Tx$.

Therefore x is a fixed point of T .

3. Interpolative Generalized Hardy-Rogers type contraction.

In this section we start by introducing the concept of interpolative G-H-R-type contraction.

Defination(3.1): Let (X,G) be a G-metric space. We say that the self-mapping $T:X \to X$ is an interpolative G-Hardy-Rogers type contraction if there exists $\lambda \in [0,1)$ and $\alpha, \beta, \gamma, \eta \in (0,1)$ with

$\alpha + \beta + \gamma + \eta < 1$, such that

$$
\begin{aligned}
G(Tx,Ty,Tz) \leq \lambda[G(x,y,z)]^{\beta}.[G(x,Tx,Tx)]^{\alpha}.[G(y,Ty,Ty)]^{\gamma}.[G(z,Tz,Tz)]^{\eta}.\\
[\tfrac{1}{2}\{G(x,Ty,Tz)+G(y,Tx,Tx)\}]^{1-\alpha-\beta-\gamma-\eta.} \qquad (3.1)
\end{aligned}
$$

for all $x, y, z \in X \setminus Fix(T) = \{u \in X,\ Tu = u\}$.

Theorem 3.1:

Let (X, G) be a Complete G-metric space and T be an interpolative G-H-R-type contraction then T has a fixed point in X.

Proof: Choose $x_0 \in X$, consider $\{x_n\}$, given as $x_n = T^n(x_0) = Tx_{n-1}$ for each positive integer n. If there exists n_0 such that $x_{n_0} = x_{n_0} + 1$, then x_{n_0} is a fixed point of T. The proof is completed.
So, assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.
By substituting the values $x = x_{n-1}$, $y = x_n$ and $z=x_n$ in (3.1), we find that

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$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \\
&\le \lambda[G(x_{n-1}, x_n, x_n)]^{\beta}.[G(x_{n-1}, Tx_{n-1}, Tx_{n-1})]^{\alpha} \\
&[G(x_n, Tx_n, Tx_n)]^{\gamma}.[G(x_n, Tx_n, Tx_n)]^{\eta}. \\
&[\frac{1}{2}\{G(x_{n-1}, Tx_n, Tx_n)+G(x_n, Tx_{n-1}, Tx_{n-1})\}]^{1-\alpha-\beta-\gamma-\eta.} \\
&\le \lambda[G(x_{n-1}, x_n, x_n)]^{\beta}.[G(x_{n-1}, x_n, x_n)]^{\alpha}. [G(x_n, x_{n+1}, x_{n+1})]^{\gamma}. \\
&[G(x_n, x_{n+1}, x_{n+1})]^{\eta}.[\frac{1}{2}G(x_{n-1}, x_{n+1}, x_{n+1})]^{1-\alpha-\beta-\gamma-\eta.}
\end{aligned}
$$

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) \le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha+\beta}.[G(x_n, x_{n+1}, x_{n+1})]^{\gamma+\eta} \\
[\frac{1}{2}\{G(x_{n-1}, x_n, x_n)+G(x_n, x_{n+1}, x_{n+1})\}]^{1-\alpha-\beta-\gamma-\eta.} \qquad (3.2)
\end{aligned}
$$

Suppose that $G(x_{n-1}, x_n, x_n) < G(x_n, x_{n+1}, x_{n+1})$ for some $n \ge 1$.

Thus, $\frac{1}{2}[G(x_{n-1}, x_n, x_n)+G(x_n, x_{n+1}, x_{n+1}) \le G(x_n, x_{n+1}, x_{n+1})$.

Consequently, the inequality (3.2) yields that

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) \le \lambda[G(x_{n-1}, x_n, x_n)]^{\beta+\gamma}.[G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha-\beta} \\
[G(x_n, x_{n+1}, x_{n+1})]^{\alpha+\beta} \le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha+\beta}. \qquad (3.3)
\end{aligned}
$$

So, we conclude that, $G(x_{n-1}, x_n, x_n) \ge G(x_n, x_{n+1}, x_{n+1})$, which is a contradiction.

Thus, we have $G(x_n, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n)$ for all $n \ge 1$.

Hence, $\{G(x_{n-1}, x_n, x_n)\}$ is a non-decreasing sequence with positive terms. Set $l = \lim_{n\to\infty} G(x_{n-1},x_n,x_n)$.

$$\frac{1}{2}[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})] \le G(x_{n-1}, x_n, x_n); \ \forall n \ge 1.$$

By a simple elimination, the inequality (3.2) implies that

$$[G(x_n, x_{n+1}, x_{n+1})]^{1-\gamma-\eta} \le \lambda[G(x_{n-1}, x_n, x_n)]^{1-\gamma-\eta} \quad \textit{for} \text{ all } n \ge 1. \qquad (3.4)$$

We deduce that

$$G(x_n, x_{n+1}, x_{n+1})] \le \lambda G(x_{n-1}, x_n, x_n) \le \lambda^n G(x_0, x_1, x_1) \qquad (3.5)$$

On account of the assumption that $\lambda<1$, by taking $n \to \infty$ in the inequality(3.4), we get that $l=0$. In what follows, we shall prove that $\{x_n\}$ is a G-Cauchy sequence by employing standard tools. More precisely, starting with the rectangular inequality, we shall get the following estimation:

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$$\begin{aligned} G(x_n, x_{n+r}, x_{n+r})] &\le G(x_n, x_{n+1}, x_{n+1}) + \dots\dots\dots + G(x_{n+r-1}, x_{n+r}, x_{n+r}) \\ &\le \lambda^n G(x_0, x_1, x_1) + \dots\dots\dots + \lambda^{n+r-1} G(x_0, x_1, x_1) \\ &\le \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_1) \qquad (3.6) \end{aligned}$$

Thus, $\{x_n\}$is a G-Cauchy sequence in the complete G-metric space(X, G), and so there exists $x \in X$ such that $\lim_{n\to\infty} G(x_n,x,x) = 0$. Suppose that $x \ne Tx$. Since $x_n \ne Tx_n$ for each $n \ge 0$, by letting $x = x_n$, $y = x$ *and* $z = x$ in (3.1), we have

$$\begin{aligned} G(x_{n+1}, Tx, Tx) = G(Tx_n, Tx, Tx) &\le \lambda[G(x_n, x, x)]^{\beta}[G(x_n, Tx_n, Tx_n)]^{\alpha}[G(x, Tx, Tx)]^{\gamma}[G(x, Tx, Tx)]^{\eta} \\ &\quad [\frac{1}{2}\{G(x_n, Tx, Tx)+G(x, Tx_n, Tx_n)\}]^{1-\alpha-\beta-\gamma-\eta} \qquad (3.7) \end{aligned}$$

letting $n \to \infty$ in the inequality (3.7), we find that $G(x, Tx, Tx)=0$, which is a contradiction.

Therefore Tx=x.

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