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Some Aspects of an Undirected Graph on a Finite Subset of Natural Numbers

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Abstract

In this paper, we investigate some special properties of the undirected graph $G_{m,n}; m,n \in \mathbb{N}$, defined in [1], on a finite subset of natural numbers. We also determine various properties of $\overline{G}_{m,n}; m,n \in \mathbb{N}$, the complement of $G_{m,n}; m,n \in \mathbb{N}$.

Keywords: connected graph, complete graph, and regular graph.
MSC 2010 Codes: 94C15, 51M05, 05C50, 15A42

1. Background

Researchers are studying about various graphs on different algebraic structures since the last many years. In 1878, Cayley [2,3] introduced an idea of constructing Cayley diagraph of a group. In 1964, Bosak [4] studied some graphs on semigroups. Zelinka [5] studied intersection graphs of nontrivial subgroups of finite abelian groups. Kelarev and Quinn [6] introduced three special graphs over semigroups-power graphs, divisibility graphs and annihilator graphs. More recently in 2015, I. Chakrabarty [1] defined an undirected graph $G_{m,n}; m,n \in \mathbb{N}$ with $V = \{1,2,3,\cdots,n\}$ and $(x,y) \in E$ if and only if $x \neq y$ and $x+y$ is not divisible by m. The graph $G_{m,n}; m,n \in \mathbb{N}$ is always connected [1] for $m,n > 1$. The graph $G_{m,n}$ [1] is isomorphic to K_3, complete $k-$partite, complete and Eulerian for certain values of m and n. I. Chakrabarty [1] also computed the degree of each vertex and the total number of edges of the graph $G_{m,n}$ and also, found the list of values of m and n for which $G_{m,n}$ is Hamiltonian.

In this present paper, we investigate some more special properties of the undirected graph $G_{m,n}; m,n \in \mathbb{N}$, defined in [1], on a finite subset of natural numbers. We also determine various properties of $\overline{G}_{m,n}; m,n \in \mathbb{N}$, the complement of $G_{m,n}; m,n \in \mathbb{N}$.

2. Preliminaries

$G = (V,E)$ is a finite, simple, undirected graph with n number of vertices and m number of edges. For $m,n \in \mathbb{N}$ and $m \neq 1$, let $G_{m,n} = (V,E)$ be a graph with $V = \{1,2,3,\cdots,n\}$ and $(x,y) \in E$ if and only if $x \neq y$ and $x+y$ is not divisible by m. For $m,n \in \mathbb{N}$ and $m \neq 1$, the complement of the graph $G_{m,n}$ is a graph $\overline{G}_{m,n} = (V,E)$, where $V = \{1,2,3,\cdots,n\}$ and $(x,y) \in E$ if and only if $x \neq y$ and $x+y$ is divisible by m. We use $d(x,y)$ to count the minimum number of edges between the vertices x and y. The following results are taken from [1] for our reference.

Theorem 2.1: The graph $G_{m,n}; m,n \in \mathbb{N}$ is always connected [1] for $m,n > 1$, except $G_{3,2}$.

Theorem 2.2: The distance between any two distinct vertices in $G_{m,n}; m,n \in \mathbb{N}$ [1] is 1 or 2.

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Theorem 2.3: For prime $p>2$, The graph $G_{p,p}$ is biregular [1] with $p-1$ vertices of degree $p-2$ and one vertex is of degree $p-1$.

3. Connectness and size of the graph $G_{m,n}; m,n \in \mathbb{N}$ and its complement $\overline{G}_{m,n}; m,n \in \mathbb{N}$

Theorem 3.1: Except $\overline{G}_{3,2}$, the graph $\overline{G}_{m,n}$ is always disconnected for $m,n \in \mathbb{N}$ and $n>1$.

Proof: The graph $G_{m,n}; m,n \in \mathbb{N}$ is always connected [1] for $m,n>1$, except $G_{1,n}$ and $G_{3,2}$. Therefore, the graph $\overline{G}_{m,n}; m,n \in \mathbb{N}$ is always disconnected for $m,n \in \mathbb{N}$ and $n>1$, except $\overline{G}_{3,2}$.

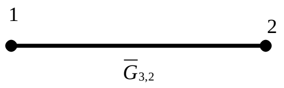

Figure 1

Theorem 3.2: The distance between any two distinct vertices in $\overline{G}_{m,n}; m,n \in \mathbb{N}$ is either 0 or 1.

Proof: The distance between any two distinct vertices in $G_{m,n}; m,n \in \mathbb{N}$ [1] is 1 or 2. If $d(i,j)=1$ in $G_{m,n}$ then $d(i,j)=0$ in $\overline{G}_{m,n}$. If $d(i,j)=2$ in $G_{m,n}$ then there exists $k \in V$ such that $d(i,k)=1$ and $d(j,k)=1$ in $G_{m,n}$ and hence $d(i,j)=1$ in $\overline{G}_{m,n}$.

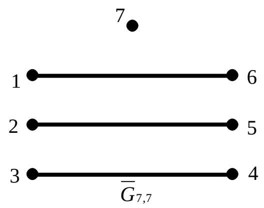

Figure 2

Corollary 3.3: For prime $p>2$, the sum of the degrees of the vertices in $G_{p,p}$ is $(p-1)^2$.

Proof: From theorem 2.3, For prime $p>2$, The graph $G_{p,p}$ is biregular with $p-1$ vertices of degree $p-2$ and one vertex is of degree $p-1$. Hence, the sum of the degrees of the vertices in $G_{p,p}$ is

$$(p-1)(p-2)+(p-1)=(p-1)^2.$$

Corollary 3.4: For prime $p>2$, the size of the graph $G_{p,p}$ is $\frac{(p-1)^2}{2}$.

Proof: From theorem 2.3, For prime $p>2$, The graph $G_{p,p}$ is biregular with $p-1$ vertices of degree $p-2$ and one vertex is of degree $p-1$. Hence, the size of $G_{p,p}$ is

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$$\frac{(p-1)(p-2)+(p-1)}{2}=\frac{(p-1)^2}{2}$$

Theorem 3.5: For prime $p>2$, the graph $\overline{G}_{p,p}$ has one component K_1 and $\frac{p-1}{2}$ components, each of them are K_2.

Proof: For prime $p>2$, consider $V=\{1,2,3,\cdots,p\}$ in $\overline{G}_{p,p}$. Clearly p does not divide $p+i, \forall i \in V, i \neq p$. So p is an isolated vertex and hence one component of $\overline{G}_{p,p}$ is K_1. For $i \in V$ and $i \neq p$, p divides $i+p-i$. Hence i and $p-i$ are adjacent. On the other hand, i and $i+1$ are not adjacent except $i=\frac{p-1}{2}$. Thus, i and $p-i$ make a K_2 for each i. Clearly, the number of such K_2 is $\frac{p-1}{2}$.

Corollary 3.6: For prime $p>2$, the graph $\overline{G}_{p,p}$ has $\frac{p+1}{2}$ components.

Proof: For prime $p>2$, the graph $\overline{G}_{p,p}$ has one component K_1 and $\frac{p-1}{2}$ components, each of them are K_2. Therefore, the total number of components is

$$\frac{p-1}{2}+1=\frac{p+1}{2}$$

Corollary 3.7: For prime $p>2$, The size of the graph $\overline{G}_{p,p}$ is $\frac{p-1}{2}$.

Proof: From theorem 3.5: For prime $p>2$, The graph $\overline{G}_{p,p}$ has exactly $\frac{p+1}{2}$ components. Out of these $\frac{p+1}{2}$ components, $\frac{p-1}{2}$ components are K_2 and one component is K_1. Each K_2 has one edge and K_1 has zero edge. Hence, the size of $\overline{G}_{p,p}$ is $\left(\frac{p-1}{2}\right)\cdot 1+0=\frac{p-1}{2}$

Theorem 3.8: For $n \geq 1$, the graph $\overline{G}_{2,2n}$ has exactly two components, each of them are K_n.

Proof: Let $V=\{1,2,3,\cdots,2n\}$ in $\overline{G}_{2,2n}$ be partitioned in $V_1=\{1,3,5,\cdots,2n-1\}$ and $V_2=\{2,4,6,\cdots,2n\}$. Clearly, 2 divides $i+j, \forall i,j \in V_x, x=1,2$ and hence, i and j are adjacent in each $E_x, x=1,2$. Also, for $i \in V_1$ and $j \in V_2$, $i+j$ is odd, not divisible by 2 and hence they are not adjacent. As a result, we have two components of $\overline{G}_{2,2n}$, each of K_n.

Corollary 3.9: For $m=2, n \geq 1$, the sum of the degrees of the vertices in $\overline{G}_{m,2n}; m,n \in \mathbb{N}$ is $2n^2-2n$.

Proof: From theorem 3.8: For $m=2, n \geq 1$, each of the two components of $\overline{G}_{m,2n}; m,n \in \mathbb{N}$ is K_n. Each K_n has n vertices and each vertex has degree $(n-1)$. The sum of the degrees is $n(n-1)+n(n-1)=2n^2-2n$.

Corollary 3.10: For $m=2, n \geq 1$, the size of the graph $\overline{G}_{m,2n}; m,n \in \mathbb{N}$ is n^2-n.

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Proof: From theorem 3.8: For $m=2, n\geq 1$, each of the two components of $\overline{G}_{m,2n}; m,n\in\mathbb{N}$ is K_n. Therefore, the size of the graph $\overline{G}_{m,2n}; m,n\in\mathbb{N}$ is

$$^{n}C_2 + {}^{n}C_2 = 2\,{}^{n}C_2 = 2\frac{n!}{(n-2)!2!} = n^2 - n.$$

Theorem 3.11: For $n\geq 1$, the graph $\overline{G}_{2,2n+1}$ has exactly two components K_{n+1} and K_n.

Proof: Let $V=\{1,2,3,\cdots,2n+1\}$ in $\overline{G}_{2,2n+1}$ be partitioned in $V_1=\{1,3,5,\cdots,2n+1\}$ and $V_2=\{2,4,6,\cdots,2n\}$ such that $|V_1|=n+1$ and $|V_2|=n$. Clearly, 2 divides $i+j, \forall i,j\in V_x, x=1,2$ and hence, i and j are adjacent in each $E_x, x=1,2$. Also, for $i\in V_1$ and $j\in V_2$, $i+j$ is odd, not divisible by 2 and hence they are not adjacent. As a result, the graph $\overline{G}_{2,2n+1}$ has exactly two components K_{n+1} and K_n.

Corollary 3.12: For $m=2, n\geq 1$, the sum of the degrees of the vertices in $\overline{G}_{m,2n+1}; m,n\in\mathbb{N}$ is n^2.

Proof: From theorem 3.11: For $m=2, n\geq 1$, the graph $\overline{G}_{m,2n+1}; m,n\in\mathbb{N}$ has exactly two components K_{n+1} and K_n. The sum of the degrees is $(n+1)n+n(n-1)=n^2$.

Corollary 3.13: For $m=2, n\geq 1$, the size of the graph $\overline{G}_{m,2n}; m,n\in\mathbb{N}$ is .

Proof: From theorem 3.11: For $m=2, n\geq 1$, the graph $\overline{G}_{m,2n+1}; m,n\in\mathbb{N}$ has exactly two components K_{n+1} and K_n. Therefore, the size of the graph $\overline{G}_{m,2n+1}; m,n\in\mathbb{N}$ is

$$^{n+1}C_2 + {}^{n}C_2 = \frac{(n+1)!}{2!(n-1)!} + \frac{n!}{2!(n-2)!} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n^2}{2}$$

Theorem 3.14: For $n\geq 1$, the $\overline{G}_{3,3n}$ has two components, K_n and an $n-$regular graph with $2n$ number of vertices.

Proof: Let $V=\{1,2,3,\cdots,3n\}$ in $\overline{G}_{3,3n}$ be partitioned in $V_1=\{3,6,9\cdots,3n\}$ and $V_2=\{1,2,4,5,7,8,\cdots,3n-1\}$ such that $|V_1|=n$ and $|V_2|=2n$. Clearly, 3 divides $i+j, \forall i,j\in V_1$ and hence, i and j are adjacent in E_1, that makes a complete graph K_n. Also, for $i\in V_2$, 3 divides $i+2i$ and hence, i and $2i$ are adjacent in E_2, that makes an $n-$regular graph with $2n$ number of vertices. On the other hand, for $i\in V_1$ and $j\in V_2$, $i+j$ is not divisible by 3.

Corollary 3.15: For $m=3, n\geq 1$, the sum of the degrees of the vertices in $\overline{G}_{m,3n}; m,n\in\mathbb{N}$ is $3n^2-n$.

Proof: From theorem 3.14, For $m=3, n\geq 1$, the $\overline{G}_{m,3n}; m,n\in\mathbb{N}$ has two components, K_n and an $n-$regular graph with $2n$ number of vertices. The sum of the degrees is

$$n(n-1)+2n\cdot n = n^2-n+2n^2 = 3n^2-n.$$

Corollary 3.16: For $m=3, n\geq 1$, the size of the graph $\overline{G}_{m,3n}; m,n\in\mathbb{N}$ is $\frac{(3n^2-n)}{2}$.

Proof: From corollary 3.15, For $m=3, n\geq 1$, the sum of the degrees of the vertices in $\overline{G}_{m,3n}; m,n\in\mathbb{N}$ is $3n^2-n$. Therefore, the size of the graph $\overline{G}_{m,2n}; m,n\in\mathbb{N}$ is $\frac{(3n^2-n)}{2}$.

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