Solution of Vacuum Field Equations Using Contraction Principle

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Abstract
Einstein field equations in general relativity have engaged number of scientists and mathematicians to explore the possibilities of finding a solution of these and to understand the nature of universe ever since their evolution. The vacuum field Einstein equation dynamics insights the same in vacuum or free space for space-time. To study all types of hyperbolic solutions globally, one of the most convincing techniques is Cauchy formulation. J. Isenberg [1997]. Whereas, other methods viz., the conformal method, constant mean curvature data (CMC-data) and non-constant mean curvature data (non-CMC-data) may also be considered in this accord. To reinvestigate the different aspects of exploring the significant proximities to obtain a solution of Einstein's vacuum field equations applying Jungck type contractions, is our main interest in this statement of paper. During this discourse, the earlier contraction mapping approach is taken into account and we will try to get a common coincident point or common fixed point as a solution for the same.

Keywords: *CMC data, Non-CMC data, Common fixed point, Jungck contraction.*

I. Introduction

In general relativity, a set of sixteen coupled non-linear partial differential equations describing the induction of gravitational effects of a body of mass, is represented by Einstein equations. These were introduced by Albert Einstein in the year 1915 as a tensor equation [1]. Further, they were associated the local space-time with local momentum and tensor in the same space-time. For a known system of stress energy, the metric tensor of the space time is determined by these equations. Thus, these equations can be expressed as a system of non-linear partial differential equations. These coupled partial differentials are hyperbolic and elliptic nonlinear equations, which become ten in numbers due to the symmetry of $G_{\alpha\beta}$ and $T_{\alpha\beta}$ along with another four special identities called Bianchi identities. These four identities satisfy $G_{\alpha\beta}$ for each single co-ordinates.

Einstein equations are of the form (Wikipedia)

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = \frac{8\pi G}{c^4}\, T_{\alpha\beta} \quad \ldots\ldots\ldots (1)$$

Where, $R_{\alpha\beta}$ is Ricci curvature tensor, R is scalar curvature, $g_{\alpha\beta}$ is metric tensor, Λ is a cosmological constant, G is the gravitational constant as proposed by Newton, c is speed of light and $T_{\alpha\beta}$ is a tensor representing stress-energy. According to Einstein, the space-time is a four dimensional manifold, called Lorentzian manifold, where the tensor field and gravity both are combined to each other and the gravity tensor is itself specified by the gravity [1]. No co-ordinate system in a Lorentzian manifold exists, except for locally flat metric, where all Christofell symbols in its domain become zero.

Thus, in General Relativity space and time are combined in the differentiable manifold provided with a pseudo-Riemannian metric g of Lorentzian signature. The length of time-like curve ascertains the intrinsic proper time along the curve. Heavy point-like object in the free fall follows a time-like geodesic. Light rays traverse the null geodesic [19].

To discuss about the solution of Einstein equations we need to specify the concerned matter-fields first. That is, in what ways the energy-momentum tensor is obtained using these field equations and what are the field equations. Then, the solution of the Einstein equations is nothing but, the solution of these coupled system of partial differential equations containing these two sets of equations. Although, due to non linearity of the field equations, it is not very easy to find an exact solution. So, certain assumptions are taken into consideration while solving these equations, which will be detailed letter. The metrics of the space-time, which furnishes the details of the structure of the space-time along with inertial motion of the body in the same space time, is the solution of Einstein's equation.

As discussed about the solution of Einstein equations, can be obtained by using the coupled system of partial differential equations in which the fixed points of the concerned dynamical system is the solution. While investigating the Einstein equations' solution, we are interested to explore the same in vacuum field, that is, in empty space, where the energy-momentum tensor vanishes so that, the field equations takes the form [18]

$R_{\alpha\beta} = 0$ ……… (2)

Here, $R_{\alpha\beta}$ is called contracted curvature tensor. Thus, in free or empty space, there will be ten field equations among which all are not independent to each and other. Some of these equations satisfy the well-known four Bianchi-identities. So, these equations remain six in numbers [18].

Cauchy formulation has been used as the most effective technique of studying the space of all globally hyperbolic solutions and to find the solution of the Einstein vacuum field equations [4] [5] [8].

In Cauchy formulation a manifold of the dimension-three along with some initial fields of data satisfying Einstein's constraint equations are assumed. With the help of freely chosen time-dependent lapse function, the required globally hyperbolic solution is obtained [Choquet-Bruhat 1962].

Finding a solution by using conformal method was first introduced by Choquet-Bruhat, York and Lichnerowitcz in around 1980. It has been the most significant tool for last forty years for finding the solution of Einstein equations. In this method, from the two parts of the split data, one is taken freely while the other is determined by using the constraint equations.

Further, in conformal method, if the data is so chosen that τ is constant then, it is dealt under constant mean curvature CMC data however, for last couple of decades the non constant mean curvature Non-CMC data, though being more rigorous, has been significantly progressed in this field [14].

In non-CMC data 4-partial differential equations along with four functions $\{\emptyset, W\}$ are considered instead of single partial differential equation with one unknown function. The key idea was to get a sequence of sol[s] $\{\emptyset_n, W_n\}$ to the constraint equations and to show its convergence to some $\{\emptyset_\infty, W_\infty\}$. Then, we have $\{\emptyset_\infty, W_\infty\}$ as the solution of the chosen group of conformal data. The uniqueness of the solution sequence is obtained by using the concept of contraction mapping. Due to following arguments we are keen to go through the concept of non constant mean curvature data while using the argument of fixed point theorem for more than single mappings.

- Before Bartnik [1988], Eardley and Witt [1989], it was speculated that every globally hyperbolic space-time with compact Cauchy surface has at least a constant mean curvature Cauchy surface but, at least for certain topologies involving Einstein equation's solution with matter, it remains no longer true [15].
- Thus, the consideration of non-constant mean curvature data (non-CMC) is impending.
- Rather, it is more difficult to solve through the said method, as in non-constant mean curvature data four PDEs and four functions $\{\emptyset, W\}$ have been taken into the account instead of one PDE with one unknown function because, $\nabla\tau \neq 0$
- Consideration of Banach space [2] along with Schauder fixed point theorem [3] has been a trend to get the solution.

- Recently, in 2015, Schaefer's fixed point theorem which is a special case of Leray- Schauder theorem has been applied to show more simple way of finding the proof with non-constant mean curvature by C. Nguyen [17].
- To show the existence and uniqueness of the solution the contraction mapping method and Banach fixed point theorem has been applied.

Thus, our key inquisitiveness in this discourse is to investigate various aspects of finding the solution of Einstein equations as well as to trace the same course of action to explore the feasibility of applying Jungck type contractions along with distinct fixed point theorem, basically common fixed point, instead of earlier established contraction mapping for the discussion to provide the solution of the famous Einstein vacuum field equations.

II. Preliminaries

Before dealing the main idea some basic definitions related to tensor calculus and general relativity are furnished here:

Definition 2.1: [19] An area of the cosmos is defined as a flat space-time if all $g_{\alpha\beta} =$ constant.

Definition 2.2: [19] A Riemannian metric is a quadratic form described by g if it is positive and definite. In short we say it as Riemannian.

Definition 2.3: [19] If g is a pseudo Riemannian with $(-,+,+,\ldots\ldots+)$ as the signature of the quadratic form then it is known as a Lorentzian metric. For example the Lorentzian metric g is given by

$g = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ ……… (3)

Definition 2.4: [19] The g trace of a curvature tensor is known as Ricci tensor with the following components.

$R_{\alpha\beta} := R^{\lambda}{}_{\boldsymbol{\lambda\alpha\beta}} \equiv \partial_{\boldsymbol{\lambda}}\Gamma^{\lambda}{}_{\boldsymbol{\alpha\beta}} - \partial_{\boldsymbol{\alpha}}\Gamma^{\lambda}{}_{\boldsymbol{\beta\lambda}} + \Gamma^{\lambda}{}_{\boldsymbol{\alpha\beta}}\Gamma^{\mu}{}_{\lambda\mu} - \Gamma^{\lambda}{}_{\boldsymbol{\alpha\mu}}\Gamma^{\mu}{}_{\beta\lambda}$ ……… (4)

Definition 2.5: [19] The g trace of Ricci tensor is called scalar curvature, R given by

$R := g^{\alpha\beta}R_{\alpha\beta}$ ……… (5)

Definition 2.6: [18] A system of coordinates x^{α} is known as geodesic system of coordinates with pole ρ_0 if all $g_{\alpha\beta}$ are locally constant in its neighborhood of the pole. Therefore,

$\frac{\partial g_{\alpha\beta}}{\partial dx^{\gamma}} = 0$, at pole ρ_0

and $\frac{\partial g_{\alpha\beta}}{\partial dx^{\gamma}} \neq 0$, otherwise.

Definition 2.7: [18] If ρ_0 is proper density of the matter and $\frac{dx^{\alpha}}{ds}$ is representing the motion of it with speed of light as unity then, the energy momentum tensor is defined as $(\rho_0 \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds})$.

Theorem 2.1: [18] For Bianchi identities in each coordinate system, we have

$R^{\alpha}{}_{ijk,i} + R^{\alpha}{}_{ijk,j} + R^{\alpha}{}_{ijk,k} = 0$ ……… (6)

Theorem 2.2: [18] A given coordinate system is geodesic coordinate if and only if all of their 2nd order covariant derivatives with respect to spatial coordinates becomes zero at the pole.

Theorem 2.3: [18] In any n-dim space there exists a geodesic system of coordinates having an arbitrary pole.

III. Materials and Methods

In Cauchy formulation, initially, a three dimensional manifold $\Sigma^3 = S^3$ along with initial data fields γ^{ab} and K^{cd} as a Riemannian metric and a symmetric tensor field on Σ^3, respectively, has been chosen. So that Einstein constraint equations

$\nabla_{\boldsymbol{a}} K^{a}{}_{\boldsymbol{b}} - \nabla_{\boldsymbol{b}} K^{c}{}_{\boldsymbol{c}} = 0$ ……… (7)

$R - K^{ab} K_{ab} (K^{c}{}_{c})^2 = 0$ ……… (8)

satisfy the initial data. Then, as per the following Einstein evolution equations the initial data fields in time are obtained [14].

$\frac{d}{dt}\gamma^{ab} = -2NK_{ab}$ ……… (9)

$\frac{d}{dt}K^{c}{}_{d} = N(R^{c}{}_{d} + K^{m}{}_{m}K^{c}{}_{d}) - \nabla^{c}\nabla_{d}N$ ……… (10)

Here, N is the time –dependent scalar function on Σ^3, called lapse function. Ultimately, on setting the following equations

$M^4 = \Sigma^3 \mathbb{R}^1$ ……… (11)

$g = -N^2 dt^2 + \gamma^{ab} dx^a dx^b$ ……… (12)

The globally hyperbolic solution of Einstein equation is obtained as (M^4, g) [4] [5] [8].

Conformal Method – [4] [5] This method was first developed by Lichnerowicz, Choquet-Bruhat and York [1969]. In this method the data (γ, K) has been split into two parts, one of which is chosen freely and other is determined by the constraint equations. The first part of the data consists of (i) a Riemannian metric λ_{ab} on Σ^3 (ii) a tensor σ_{cd} on Σ^3 with no divergence $\nabla_c \sigma^{cd} = 0$ and no trace $\lambda_{cd}\sigma^{cd} = 0$ and (iii) a scalar field τ on Σ^3. The second part of the data consists of (i) a vector field W^bon Σ^3 and (ii) a definite positive scalar field φ on Σ^3. To get the solution using the conformal method, the conformal data (λ, σ, τ) is considered and then, the following equations are solved for W and φ.

$\nabla^2 \emptyset = \frac{1}{8} R\emptyset - \frac{1}{8}\left(\sigma^{ab} + LW^{ab}\right)(\sigma_{ab} + LW_{ab})\emptyset^{-7} + \frac{1}{12}\tau^2\emptyset^5$ ……… (13)

$\nabla_a (LW)^{a}{}_{b} = \frac{2}{3}\emptyset^6 \nabla_b \tau$ ……… (14)

Where, ∇ is covariant derivative compatible with λ, R is the scalar curvature and L is the conformal killing operator such that,

$LW_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{3}\lambda_{ab}\nabla_c W^c$ ……… (15)

Thus, for the obtained set $(\emptyset, W)$

$\gamma_{ab} = \emptyset^4 \lambda_{ab}$ ……… (16)

$K^{cd} = \emptyset^{-10}\left(\sigma^{cd} + LW^{cd}\right) + \frac{1}{3}\emptyset^{-4}\lambda^{cd}\tau$ ……… (17)

will be a required solution of the said constraint equations.

Constant Mean Curvature Data – [14] [15] It is more simple because, here, $\nabla\tau = 0$ condition transforms the constraint equations into a single equation which takes Lichnerowicz form

$\nabla^2 \emptyset = \frac{1}{8} R\emptyset - \frac{1}{8}\sigma^2\emptyset^{-7} + \frac{1}{12}\tau^2\emptyset^5$ ……… (18)

with $$\nabla_a (LW)^a{}_b = 0 \Rightarrow (LW)^a{}_b = 0 \ldots\ldots\ldots (19)$$

It can easily be solved for one function $\emptyset$.

Non-Constant Mean Curvature Data – [14] [15] [17] In non-CMC data the condition $\nabla\tau \neq 0$ makes the situation so complicated. But, in short it may be understood that the following path has been traced to obtain the required solution.

- The main idea was to establish a sequence of solutions $\{\emptyset_n, W_n\}$ to constraint equations and its convergence to some $\{\emptyset_\infty, W_\infty\}$ is obtained.
- Then, $\{\emptyset_\infty, W_\infty\}$ is a solution of the selected set of conformal data.
- To show the convergence of the said sequence, existence of n-independent upper and lower bounds for the supposed sequence has been established.
- The argument of contraction mapping is then used and uniqueness of the acquired limit of the sequence is established to get the required solution.

Jungck Contraction in Non-CMC
Our approach of investigation of the well posed problem can be perceived through the following theorem in which two mappings have been considered in place of single contraction mapping as passed earlier in ref. [14] [17], etc.
Theorem 3.1: [6] [7] [13] For any two self mappings I and J on a complete metric (S, b) space for which, we have

1. (I, J) is commuting
2. J is continuous
3. $I(S)$ is the subset of $J(S)$.
4. Then, $\exists\ \mu \in [0,1)$ such that,

$$b(I_x, I_y) \leq \mu b(J_x, J_y)$$

Then, I and J have a unique common fixed point. .
So far as non CMC data is concerned, with the suitable choices of the two functions out of the four functions $\{\emptyset, W\}$ as discussed earlier, it may be possible to apply Jungck contraction to have a unique common fixed point by making use of the convergence of the concerned Cauchy sequences as dealt earlier.

IV. Conclusion
Thus, it may easily be observed that to understand the relation between the gravitational field and matter, Einstein equations have played a vital role for decades, which will be reconciled by making use of their solution, for which number of physicists and scientists were engaged ever since. Till now, various methods to solve these equations were implemented by different workers of the field. In this context, the role of fixed point theorems, based on the principle of contraction, have not been ignored. Here, we do suggest some thought to achieve the same by using two functions among the four found in non-CMC data and getting unique common fixed point by making use of Jungck type contractions.

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