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Fuzzy solutions for wave equation with fuzzy coefficient

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Abstract

In this study we investigate wave equation as classical model of partial differential equations (PDE) with uncertain parameters. Our approach is based on employing the concept of cross product of fuzzy numbers. Using this concept, we construct the triangular fuzzy solutions. Finally, we compare the uncertainty of the solutions

Keywords: *Fuzzy partial differential equation, Wave equation, The cross product of fuzzy numbers, Triangular fuzzy numbers.*

1. Introduction

Differential equations play an important role in the modeling of physical and engineering problems, such as solid, fluid mechanics, biology and physics. In general, the parameters, variables and initial conditions within a model are considered as being defined exactly. In reality there may be only vague, imprecise or incomplete information about the variables and parameters available. These problems are modeled by fuzzy partial or ordinary differential equations. The fuzzy differential calculus is developed by different authors, like Dubois (1982), Puri and Ralescu (1983), Kaleva (1987) and Seikkala (1987). The theory of fuzzy differential equations with different types of differentiability concept is extensively studied. As a related precedent, we may mention [5, 9, 10, 11, 12, 13, 16, 18, 22, 24].

It is well known that many phenomena of nature or physical systems are modeled by partial differential equations (PDEs), such as wave equations, heat equations, and so on. Hence, studies of PDEs have become one of the main topics of modern mathematical analysis and have attracted much attention. However, the exact solutions to the PDEs cannot be easily obtained except for very simple or special cases.

Fuzzy partial differential equations were first introduced by Buckley and Feuring in [15]. In recent years, many methods have been developed for solving some kinds of PDEs. For example in [3], the authors proposed difference method for solving FPDEs. Later the authors in [20] have been applied fuzzy logic and appropriate rule-based systems to construct the solutions for fuzzy partial differential equations which model the reoccupation of ants in a region of attraction. Arshad and Lupulescu have studied the existence and uniqueness of the solution of a class of fractional differential equation with fuzzy initial value in [6]. Bertone and Jafelice have employed the Zadeh’s extension principle to obtain these solutions in [14]. The authors [4] have studied the existence and uniqueness of the solution of the fuzzy heat equation based on generalized Hukuhara partial derivatives and obtained their analytical solutions. In [7], Bahrami et al. have studied the linear first order fuzzy transport equation in homogeneous and nonhomogeneous cases with fuzzy initial condition by using generalized Hukuhara differentiability and presented the solution when speed parameter is a fuzzy number. Alikhani et al. have studied the linear first order fuzzy transport equation under the cross product of fuzzy number in [2]. The authors in obtained some new results on the directional derivative of the summations and H-difference of fuzzy multi-variable functions and have studied the wave equation with fuzzy initial values based on the concept of strongly generalized derivatives [1]. The authors obtained the solution of a fuzzy heat equation by using fuzzy Fourier transform in [19]. Long et al. have investigated the existence and uniqueness of fuzzy solutions for hyperbolic partial differential equations in [23].

In this study, we make use of the concept of cross product for fuzzy numbers instead of usual fuzzy products to interpret the solution concept for fuzzy partial differential equations. We investigate the fuzzy wave equation for which the speed is a fuzzy number and initial values are fuzzy number valued functions. We obtain analytical solutions of fuzzy wave equation with this approach such that the results are applicable to engineering and applied sciences and we describe our approach with an illustrative example. Finally, we compare the measure of uncertainty of solutions in wave equation problems with different conditions.

2. Preliminaries

We begin this section with definitions and lemmas that we use in this paper. The space of fuzzy numbers is denoted by R_F. For $\alpha \in [0,1]$ α -level set of $u \in R_F$ is defined by $[u]_\alpha = \{x \in R, u(x) \geq \alpha\}$, $[u]_0 = cl\{x \in R, u(x) \geq 0\}$. For any $\alpha \in [0,1]$, $[u]_\alpha = [\,\underline{u}_\alpha, \overline{u}_\alpha]$ is a bounded closed interval. The 1-level set is called the core of the fuzzy number, while the 0-level set is called the support of the fuzzy number.

Recall that a fuzzy number u is triangular if $[u]_\alpha = [u_l + (u_c - u_l)\alpha, u_r + (u_c - u_r)\alpha], 0 \leq \alpha \leq 1$, for real numbers $u_l < u_c < u_r$, where we henceforth denote it by $u = (u_l, u_c, u_r)$. The space of triangular fuzzy numbers is denoted byR_τ. For $u, v \in R_F, \lambda \in R$we define the sum $u + v$ and the product λu as $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda [u]_\alpha$, where $[u]_\alpha + [v]_\alpha$and $\lambda [u]_\alpha$ mean the usual addition of two intervals of R and the usual product between a scalar and an interval of R respectively.

In the special case, when $u = (u_l, u_c, u_r)$ and $v = (v_l, v_c, v_r)$ be two triangular fuzzy numbers and $\lambda \in R$, we define the addition as $u + v = (u_l + v_l, u_c + v_c, u_r + v_r)$ and the scalar multiplication as

$$\lambda u = \begin{cases} (\lambda u_l, \lambda u_c, \lambda u_r) & \lambda \geq 0 \\ (\lambda u_r, \lambda u_c, \lambda u_l) & \lambda < 0 \,. \end{cases}$$

We notice that a real number is a triangular fuzzy number for which $u_l = u_c = u_r$.

If $u \in R_F$, then we have its length as

$$len[u]_\alpha = \overline{u}_\alpha - \underline{u}_\alpha \,, \qquad \forall \alpha \in [0,1].$$

In the special case $\alpha = 0$, we denote $len([u]_0) = diamu$.

In the case that $u \in R_\tau$we have its length as

$$\mathrm{len}[u]_\alpha = (u_r - u_l) - \alpha(u_r - u_l).$$

Definition 2.1 ([10]). Let $u, v \in R_F$. If there exists a unique fuzzy number $w \in R_F$ such that $u = v + w$, then w is called the Hukuhara difference of u,v and is denoted by $u \ominus v$.

Remark 2.2 Let $u = (u_l, u_c, u_r), v = (v_l, v_c, v_r)$ be two triangular fuzzy numbers. The H-difference $u \ominus v$ exists if

$$u \ominus v = (u_l - v_l, u_c - v_c, u_r - v_r),$$

defines a triangular fuzzy number.

Lemma 2.3 *([13]). Let* $u, v \in R_\tau$ *be such that* $u_c - u_l > 0, u_r - u_c > 0$ *and* $\mathrm{len}([v]_0) = (v_r - v_l) \leq \min\{u_c - u_l, u_r - u_c\}$.

Then the H-difference $u \ominus v$ exists.

Definition 2.4 ([10]). Hausdorff distance between u and v is given by $D: R_F \times R_F \to R_+ \cup \{0\}$

$$D(u, v) = \sup_{\alpha \in [0,1]} max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u_\alpha} - \overline{v_\alpha}|\}.$$

The space (R_F, D) is a complete metric space.

Remark 2.5 To simplify the sentences, we will here after recall a triangular fuzzy number-valued, function $f: (a, b) \to R_\tau$ by a triangular fuzzy function.

Definition 2.6 Let $f: (a, b) \to R_\tau$ be as $f(x) = (f_l(x), f_c(x), f_r(x))$ and $x \in (a, b)$. We define

$$\int_a^b f(x)dx = \left(\int_a^b f_l(x)dx \,, \int_a^b f_c(x)dx \,, \int_a^b f_r(x)dx\right)$$

Lemma 2.7 ([10]). If $f: (a, b) \to R_\tau$ is integrable and $c \in (a, b)$, then

$$\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt$$

Lemma 2.8 ([10]). Let $f:(a,b)\to R_\tau$ be as $f(x)=(f_l(x),f_c(x),f_r(x))$ such that $f_i; i=l,c,r$ is real-valued differentiable function on (a, b). Then f is (i)-differentiable at $x_0\in(a,b)$ if and only if

$$f'(x_0)=\left(f_l'(x_0),f_c'(x_0),f_r'(x_0)\right),$$

defines a triangular fuzzy number. Similarly, f is (ii)-differentiable at x_0 if and only if

$$f'(x_0)=\left(f_r'(x_0),f_c'(x_0),f_l'(x_0)\right),$$

is a triangular fuzzy number. If f is (i)- or (ii)-differentiable on the entire of interval (a,b), we say that f is generalized differentiable on (a,b).

Remark 2.9 For the function f(x) the second generalized derivative is simply the generalized derivative of f'(x) (as stated in Lemma 2.8) on (a,b) and we say f is twice generalized differentiable at $x\in(a,b)$.

Lemma 2.10 ([10]). Let $f:(a,b)\to R_\tau$ be a continuous triangular fuzzy function. Then

1. $F(x)=\int_a^x f(t)dt$ is (i)-differentiable and $F'(x)=f(x)$.
2. $F(x)=\int_x^b f(t)dt$ is (ii)-differentiable and $F'(x)=-f(x)$

Lemma 2.11 ([10, 13]). Let $f,g:(a,b)\to R_\tau$ be generalized differentiable on (a,b).

1. If f and g are (i)-differentiable on (a,b), then$f+g$ is (i)-differentiable and
$$(f+g)'(x)=f'(x)+g(x),\qquad \forall x\in(a,b).$$
2. If f and g are (ii)-differentiable on (a,b), then $f+g$ is (ii)-differentiable and
$$(f+g)'(x)=f'(x)+g(x),\qquad \forall x\in(a,b).$$
3. If f is (i)-differentiable, g is (ii)-differentiable on (a,b) and the H-difference $f(x)\ominus g(x)$ exists for $x\in(a,b)$, then $f\ominus g$ is (i)-differentiable and
$$(f\ominus g)'(x)=f'(x)+(-1)g'(x),\qquad \forall x\in(a,b).$$

4.If f is (ii)-differentiable, g is (i)-differentiable on (a,b) and the H-difference $f(x)\ominus g(x)$ exists for $x\in(a,b)$,

then $f\ominus g$ is (ii)-differentiable and
$$(f\ominus g)'(x)=f'(x)+(-1)g'(x),\qquad \forall x\in(a,b).$$

Lemma 2.12 ([1]). Let $f,g:(a,b)\to R_\tau$ be generalized differentiable such that f is (i)-differentiable and g is (ii)-differentiable on (a,b).

1. If the H-difference $f'(x)\ominus(-1)g'(x)$ exists for all $x\in(a,b)$, then $f+g$ is (i)-differentiable and
$$(f+g)'(x)=f'(x)\ominus(-1)g'(x),\qquad \forall x\in(a,b).$$
2. If the H-difference $g'(x)\ominus(-1)f'(x)$ exists for all $x\in(a,b)$, then $f+g$ is (ii)-differentiable and
$$(f+g)'(x)=g'(x)\ominus(-1)f'(x),\qquad \forall x\in(a,b).$$

Lemma 2.13 ([1]). Let $f,g:(a,b)\to R_\tau$ be generalized differentiable such that f,g are (i)-differentiable on (a, b).

1. If the H-differences $f(x)\ominus g(x)$, $f'(x)\ominus g'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is (i)-differentiable and
$$(f\ominus g)'(x)=f'(x)\ominus g'(x),\qquad \forall x\in(a,b).$$
2. If the H-differences $f(x)\ominus g(x)$, $g'(x)\ominus f'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is (ii)-differentiable and
$$(f\ominus g)'(x)=(-1)(g'(x)\ominus f'(x)),\qquad \forall x\in(a,b).$$

Lemma 2.14 ([1]). Let $f,g:(a,b)\to R_\tau$ be generalized differentiable such that f,g are (ii)-differentiable on (a,b).

1. If the H-differences $f(x)\ominus g(x)$, $g'(x)\ominus f'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is (i)-differentiable and
$$(f\ominus g)'(x)=(-1)(g'(x)\ominus f'(x)),\qquad \forall x\in(a,b).$$
2. If the H-differences $f(x)\ominus g(x)$,$f'(x)\ominus g'(x)$ exist for all $x\in(a,b)$, then $f\ominus$

g is (ii)-differentiable and

$(f \ominus g)'(x) = f'(x) \ominus g'(x)), \quad \forall x \in (a,b).$

Lemma 2.15 ([10, 13]). Let $g: (a,b) \to R$ be differentiable $k \in R_F$.

1. If $g(x)g'(x) \geq 0$ for all $x \in (a,b)$, then kg is (i)-differentiable and

$(kg)'(x) = k\, g'(x), \quad \forall x \in (a,b).$

2. If $g(x)g'(x) < 0$ for all $x \in (a,b)$, then kg is (ii)-differentiable and

$(kg)'(x) = kg'(x), \quad \forall x \in (a,b).$

Lemma 2.16 ([10, 13]). Let $f: (a,b) \to R_\tau$ be generalized differentiable and $g: (a,b) \to R \;\; be\ differentiable.$

1. If f is (i)-differentiable and $g'(x) > 0$ for all $x \in (a,b)$, then $fo\ g$ is (i)-differentiable and

$(fog)'(x) = g'(x)f'(g(x)), \quad \forall x \in (a,b).$

2. If f is *(i)*-differentiable and $g'(x)$< *0* for all $x \in (a,b)$, then $fo\ g$ is **(**ii**)**-differentiable and
$(fog)'(x) = g'(x)f'(g(x)), \quad \forall x \in (a,b).$

3. If f is **(**ii**)**-differentiable and $g'(x)$> *0* for all $x \in (a,b)$, then $fo\ g$ is **(**ii**)**-differentiable and
$(fog)'(x) = g'(x)f'(g(x)), \quad \forall x \in (a,b).$
4. If f is **(**ii**)**-differentiable and $g'(x)$ < *0* for all $x \in (a,b)$, then $fo\ g$ is **(**i**)**-differentiable and
$(fog)'(x) = g'(x)f'(g(x)), \quad \forall x \in (a,b).$

Definition 2.17 Let $u = (u_l, u_c, u_r)$ be a triangular fuzzy number. We say u is positive if $u_c > 0$ and is negative if $u_c < 0$. The set of positive (negative) triangular fuzzy numbers is denoted by $R_\tau^+ (R_\tau^-)$.

Definition 2.18 ([8]). Let $u = (u_l, u_c, u_r)$and $v = (v_l, v_c, v_r)$ be two positive triangular fuzzy numbers. We denote the cross product of u and v by $u \odot v$ where

$u \odot v = (u_l v_c + v_l u_c - u_c v_c, u_c v_c, u_r v_c + v_r u_c - u_c v_c).$

Lemma 2.19 ([8]). The cross product of two positive triangular fuzzy numbers is a positive triangular fuzzy number.

In what follows, we define fuzzy two-variable function and its partial derivatives.

Definition 2.20 A triangular fuzzy two-variable function $f: \mathbb{J} \to R_\tau$ is defined by

$f(x,y) = \big(f_l(x,y), f_c(x,y), f_r(x,y)\big),$

where $\mathbb{J} \subseteq R^2$, f_l, f_c, f_r are real functions and $f_l \leq f_c \leq f_r$ for all $(x,y) \in \mathbb{J}$.

Definition 2.21 Let f be a triangular fuzzy two-variable function with $f(x,y) = \big(f_l(x,y), f_c(x,y), f_r(x,y)\big)$ such that $f_i; i = l,c,r$ is real-valued differentiable function and $(x,y) \in \mathbb{J}$. Then f is **(**i**)**-partial differentiable w.r.t. x at (x_0, y_0) if and only if

$\frac{\partial f}{\partial x}(x_0, y_0) = (\frac{\partial f_l}{\partial x}(x_0, y_0), \frac{\partial f_c}{\partial x}(x_0, y_0), \frac{\partial f_r}{\partial x}(x_0, y_0)),$

defines a triangular fuzzy number. It is obtained by keeping y fixed $y = y_0$ and finding the **(**i**)**-derivative at x_0 of the function $f\,(x, y_0)$ by Lemma 2.8. Similarly, f is **(**ii**)**-partial differentiable w.r.t. x at (x_0, y_0) if and only if

$\frac{\partial f}{\partial x}(x_0, y_0) = (\frac{\partial f_r}{\partial x}(x_0, y_0), \frac{\partial f_c}{\partial x}(x_0, y_0), \frac{\partial f_l}{\partial x}(x_0, y_0)),$

is a triangular fuzzy number. Moreover, in a similar way, we can define that f is **(**i**)**- or **(**ii**)**-partial differentiable w.r.t. y at (x_0, y_0). If f is **(**i**)**- or **(**ii**)**-partial differentiable on the entire of $\mathbb{J}$, we say that f is generalized partial differentiable.

For the function $f(x, y)$ the"own" second generalized partial derivative w.r.t. *x* is simply the generalized partial derivative of $\frac{\partial f}{\partial x}(x,y)$ w.r.t. *x* on $\mathbb{J}$, i.e.

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f_x}{\partial x} = f_{xx}.$

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}$ are (i)- or (ii)-partial differentiable w.r.t. *x,y* on the entire of $\mathbb{J}$, we say that f is twice generalized partial differentiable.

3. Homogeneous fuzzy wave equation

In this section, we study the following homogeneous fuzzy wave equation with fuzzy speed

$$u_{tt}(x,t) = k \odot u_{xx}(x,t) \qquad (x,t) \in R \times (0,+\infty) \qquad (3.1)$$
$$u(x,0) = f(x) \qquad x \in R,$$
$$u_t(x,0) = g(x) \qquad x \in R,$$

where $f,g: R \to R_\tau$ are triangular fuzzy functions and $k \in R_\tau^+$. Without, loss the generality, we let $k = (k_l, k_c^2, k_r)$, with $k_c > 0$.

Definition 3.1 We say u: $R \times (0,+\infty) \to R_\tau$ is a solution of Problem (3.1) if u is twice generalized partial differentiable w.r.t. x,t respectively and satisfies Problem (3.1) in $R \times (0,+\infty)$.

Note. To simplify the presentation, we denote the following notations.

$$v_1(x,t) = {}^1/_2 \left(f(x+k_ct) + f(x-k_ct)\right) + 1/2k_c \int_{x-k_ct}^{x+k_ct} g(s)ds,$$

$$w(x,t) = {}^1/_{2k_c} (k_l - k_c^2, 0, k_r - k_c^2) \int_0^t \int_{x-k_c(t-s)}^{x+k_c(t-s)} {}^1/_2 (f_c''(y+k_cs) + f_c''(y-k_cs))dyds,$$

$$v_2(x,t) = {}^1/_2 \left(f(x+k_ct) + f(x-k_ct)\right) \ominus (-1)\, {}^1/_{2k_c} \int_{x-k_ct}^{x+k_ct} g(s)ds$$

Moreover, we consider

$$u_1(x,t) = v_1(x,t) + w(x,t)$$
$$u_2(x,t) = v_2(x,t) \ominus (-1)w(x,t),$$

Provided the H-difference above exist.

The next theorems present main results of this section and give two different forms of fuzzy solutions of Problem (1).

Theorem 3.2 Let the triangular fuzzy functions f,g be twice generalized differentiable such that g_l'(x) is decreasing, $g_c'(x) = 0$ and g_r'(x) is increasing. Moreover, suppose that the real function f_c be differentiable of the 4th-order where the function $f_c''(x) > 0$ and the real function $f_c^{(j)} f_c^{(j+1)}$ is positive for j = *1, 2, 3.* Then

1. If f, f' and g, g' are (i)-differentiable, then u_l is a solution of Problem (1).
2. If f is (ii)-differentiable and f', g and g' are (i)-differentiable, then u_2 is a solution of Problem (1).

Theorem 3.3 Let the triangular fuzzy functions f,g be twice generalized differentiable such that g_l'(x) is decreasing, $g_c'(x) = 0$ and g_r'(x) is increasing. Moreover, suppose that the read function f_c be differentiable of the 4th-order where the function $f_c''(x) > 0$ and the real function $f_c^{(j)} f_c^{(j+1)}$is negative for j = *1, 2, 3.* Then

1. If f, f' and g, g' are (ii)-differentiable, then u_l is a solution of Problem (1).
2. If f is (i)-differentiable and f', g and g' are (ii)-differentiable, then u_2 is a solution of Problem (1).

In order to prove Theorems 2 and 3 we required the following lemmas.

Lemma 3.4 Let h: $R \to R^+$ be a differentiable function of the 2-th order. For $(x,t) \in R \times (0,+\infty)$ consider

$$p(x,t) = \int_0^t \int_{x-k_c(t-s)}^{x+k_c(t-s)} \left(h(y+k_cs) + h(y-k_cs)\right) dyds$$

Then

$\frac{\partial p}{\partial x}(x,t) = t(h(x+k_c t) - h(x-k_c t)).$

$\frac{\partial p}{\partial t}(x,t) = k_c t\big(h(x+k_c t) + h(x-k_c t)\big) + k_c \int_0^t (h(x+k_c(t-2s)) + h(x-k_c(t-2s)))ds$

$\frac{\partial^2 p}{\partial x^2}(x,t) = t\big(h'(x+k_c t) - h'(x-k_c t)\big).$

$\frac{\partial^2 p}{\partial t^2}(x,t) = 2k_c\big(h(x+k_c t) + h(x-k_c t)\big) + k_c^2 t\big(h'(x+k_c t) - h'(x-k_c t)\big).$

Moreover,

1. If the real function $h^{(j)}h^{(j+1)}$ for $j = 0,1,2$ is positive, then the functions $p, \frac{\partial p}{\partial x}, \frac{\partial^2 p}{\partial x^2}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial t^2}$ are positive.

2. If the real function $h^{(j)}h^{(j+1)}$ for $j = 0,1,2$ is negative, then the functions $p, \frac{\partial^2 p}{\partial x^2}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial t^2}$ are positive and $\frac{\partial p}{\partial x}$ is negative.

Proof. First, we compute partial derivatives of $p(x,t)$ w.r.t. *x,t* to the 2th-order. The partial derivative of $p(x,t)$ w.r.t. *x* is

$\frac{\partial p}{\partial x}(x,t) = \int_0^t (h(x+k_c t) + h\big(x+k_c(t-2s)\big) - h\big(x-k_c(t-2s)\big) - h(x-k_c t)\big)ds = t\big(h(x+k_c t) - h(x-k_c t)\big).$

By simple computation we have

$\frac{\partial^2 p}{\partial x^2}(x,t) = t\left(h'(x+k_c t) - h'(x-k_c t)\right).$

Now let us compute the partial derivative of $p(x,t)$ w.r.t. t, we obtain

$\frac{\partial p}{\partial t}(x,t) = k_c \int_0^t (h(x+k_c t) + h\big(x+k_c(t-2s)\big) + h\big(x-k_c(t-2s)\big) + h(x-k_c t)\big)ds = k_c t\big(h(x+k_c t) + h(x-k_c t)\big) + k_c \int_0^t (h\big(x+k_c(t-2s)\big) + h(x-k_c(t-2s)))ds.$

Moreover, the second partial derivative of *p(x,t)* w.r.t. *t* is

$\frac{\partial^2 p}{\partial t^2}(x,t) = k_c(h(x+k_c t) + h(x-k_c t)) + k_c^2 t(h'(x+k_c t) - h'(x-k_c t)) + k_c \frac{\partial}{\partial t}(\int_0^t (h\big(x+k_c(t-2s)\big) + h\big(x-k_c(t-2s)\big))ds) = 2k_c\big(h(x+k_c t) + h(x-k_c t)\big) + k_c^2 t\big(h'(x+k_c t) - h'(x-k_c t)\big).$

In what follows, we only give a proof for a Case. The other Cases can be investigated in a similar way and we omit the details.

Let $h^{(j)}h^{(j+1)}$ for $j = 0,1,2$ be positive and the real function h be positive. Therefore, the real functions h', h'' are positive and increasing. It follows that the results of the Case 1 are satisfied. □

Lemma 3.5 Let $f: R \to R_\tau$ be a generalized differentiable function on *R* and c is a positive constant real number. For $(x,t) \in R \times (0,+\infty)$ consider

$H_1(x,t) = f(x+ct) + f(x-ct).$

Then

1. H_1 is always generalized partial differentiable w.r.t. x such that the type of generalized partial differentiability H_1 and f w.r.t. x are the same and

$$\frac{\partial H_1}{\partial x} = f'(x+ct) + f'(x-ct).$$

2. If f, f' are (i)-differentiable or f is (ii)-differentiable and f' is (i)-differentiable, then H_1 is (i)- or (ii)-partial differentiable w.r.t. t respectively and

$$\frac{\partial H_1}{\partial t} = c(f'(x+ct) \ominus f'(x-ct)).$$

3. If f, f' are (ii)-differentiable or f is (i)-differentiable and f' is (ii)-differentiable, then H_1 is (i)- or (ii)-partial differentiable w.r.t. t respectively and

$$\frac{\partial H_1}{\partial t} = -c(f'(x-ct) \ominus f'(x+ct)).$$

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Proof. We only give a proof for a Case. The other Cases can be investigated in a similar way and we omit the details.

Let f be (i)-differentiable and f' be (ii)-differentiable. We claim that the H-difference $f'(x-ct) \ominus f'(x+ct)$ exists. Since f' is (ii)-differentiable, then $f_l'(x)-f_c'(x)$ and $f_c'(x)-f_r'(x)$are increasing for all $x \in R$. It means that for $(\mathrm{x},\mathrm{t}) \in \mathrm{R} \times (0,+\infty)$

$$\left(f_l'(x-ct)-f_l'(x+ct), f_c'(x-ct)-f_c'(x+ct), f_r'(x-ct)-f_r'(x+ct)\right) = f'(x-ct) \ominus f'(x+ct),$$

is a triangular fuzzy function. Therefore, by recalling Lemma 2.12, we conclude that, H_1 is (ii)-partial differentiable w.r.t. t and

$$\frac{\partial H_1}{\partial t} = -c(f'(x-ct) \ominus f'(x+ct)),$$

This completes the proof. □

Lemma 3.6 Let $f: R \to R_\tau$ be a generalized differentiable function on R and c is a positive constant real number. For $(x,t) \in R \times (0,+\infty)$ consider

$$H_2(x,t) = f(x+ct) \ominus f(x-ct),$$

provided the H-difference above exist on $R \times (0,+\infty)$. Then

1. If f, f' are (i)-differentiable or f is (ii)-differentiable and f' is (i)-differentiable, then H_2 is (i)-or (ii)-partial differentiable w.r.t. x respectively and

$$\frac{\partial H_2}{\partial x} = f'(x+ct) \ominus f'(x-ct),$$

2. If $f, f'$are(ii)-differentiable or f is (i)-differentiable and f' is (ii)-differentiable, then H_2 is (i)-or (ii)-partial differentiable w.r.t. x respectively and

$$\frac{\partial H_2}{\partial x} = (-1)(f'(x-ct) \ominus f'(x+ct)),$$

3. H_2 is always generalized partial differentiable w.r.t. t such that the type of generalized partial differentiability H_2 and f w.r.t. x are the same and

$$\frac{\partial H_2}{\partial t} = c(f'(x+ct) + f'(x-ct)).$$

Lemma 3.7 Let $f: R \to R_\tau$ be a generalized differentiable function on R and c is a positive constant real number. For $(x,t) \in R \times (0,+\infty)$ consider

$$H_3(x,t) = f(x-ct) \ominus f(x+ct),$$

provided the H-difference above exist on $R \times (0,+\infty)$. Then

1. If f, f' are (i)-differentiable or f is (ii)-differentiable and f' is (i)-differentiable, then H_3 is (ii)-or (i)-partial differentiable w.r.t. x respectively and

$$\frac{\partial H_3}{\partial x} = (-1)(f'(x+ct) \ominus f'(x-ct)).$$

2. If $f, f'$are(ii)-differentiable or f is (i)-differentiable and f' is (ii)-differentiable, then H_3 is (ii)-or (i)-partial differentiable w.r.t. x respectively and

$$\frac{\partial H_3}{\partial x} = f'(x-ct) \ominus f'(x+ct).$$

3. H_3 is always generalized partial differentiable w.r.t. t such that the type of generalized partial differentiability H_3 and f w.r.t. x are the same and

$$\frac{\partial H_3}{\partial t} = (-c)(f'(x+ct) + f'(x-ct)).$$

Lemma 3.8 Let $g: R \to R_\tau$ be a generalized differentiable function on R and c is a positive constant real number. For $(x,t) \in R \times (0,+\infty)$ consider

$$G(x,t) = \int_{x-ct}^{x+ct} g(s)ds.$$

Then

1. If g is (i)-differentiable, then G is (i)-partial differentiable w.r.t. x and

$$\frac{\partial G}{\partial x}(x,t) = g(x+ct) \ominus g(x-ct).$$

2. If g is (ii)-differentiable, then G is (ii)-partial differentiable w.r.t. x and

$$\frac{\partial G}{\partial x}(x,t) = (-1)(g(x-ct) \ominus g(x+ct)).$$

3. G is always (i)- partial differentiable w.r.t. t and

$$\frac{\partial G}{\partial t}(x,t) = c(g(x+ct) + g(x-ct)).$$

Proof of Theorem 3.2. Case 1. Using Lemma 3.4, we conclude that the functions p, $\frac{\partial p}{\partial x}, \frac{\partial^2 p}{\partial x^2}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial t^2}$ are positive.

Therefore, by applying lemma 2.15, we conclude that w is (i)-partial differentiable w.r.t. *x*,*t* and $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}$ are (i)-partial differentiable w.r.t. *x*,*t* respectively.

On the other hand, by recalling Lemma 3.5, we conclude that $f(x+k_c t)+f(x-k_c t)$ is (i)-partial differentiable w.r.t. x. It follows from Lemma 3.8 that the function $\int_{x-k_c t}^{x+k_c t} g(s)ds$ is (i)-partial differentiable w.r.t. x. Employing Lemma 2.11, we deduce that the function v_1 is (i)-partial differentiable w.r.t. *x* and

$$\frac{\partial v_1}{\partial x}(x,t) = {}^1\!/_2\left(f'(x+k_c t)+f'(x-k_c t)\right) + {}^1\!/_{2k_c}\left(g(x+k_c t) \ominus g(x-k_c t)\right).$$

So, u_1 is (i)-partial differentiable w.r.t. *x* and we obtain

$$\frac{\partial u_1}{\partial x}(x,t) = \frac{\partial v_1}{\partial x}(x,t) + \frac{\partial w}{\partial x}(x,t).$$

It follows from Lemma 3.5 that the function $f'(x+k_c t)+f'(x-k_c t)$ is (i)-partial differentiable w.r.t. x. From Lemma 3.6 we conclude that $g(x+k_c t) \ominus g(x-k_c t)$ is (i)-partial differentiable w.r.t. x. It means that the function $\frac{\partial v_1}{\partial x}$ is (i)-partial differentiable w.r.t. *x* and we have

$$\frac{\partial^2 v_1}{\partial x^2}(x,t) = {}^1\!/_2\left(f''(x+k_c t)+f''(x-k_c t)\right) + {}^1\!/_{2k_c}\left(g'(x+k_c t) \ominus g'(x-k_c t)\right).$$

Therefore, by the above reasoning and Lemma 2.11, we conclude that $\frac{\partial u_1}{\partial x}$ is (i)-partial differentiable w.r.t. *x* and we deduce

$$\frac{\partial^2 u_1}{\partial x^2}(x,t) = \frac{\partial^2 v_1}{\partial x^2}(x,t) + \frac{\partial^2 w}{\partial x^2}(x,t).$$

Now let us investigate partial derivative of u_1 w.r.t. t.

It follows from Lemma 3.5 that the function $f(x+k_c t)+f(x-k_c t)$ is (i)-partial differentiable w.r.t. t. We deduce Lemma 3.8, the function $\int_{x-k_c t}^{x+k_c t} g(s)ds$ is (i)-partial differentiable w.r.t. t.

It means that, v_1 is (i)-partial differentiable w.r.t. *t* and we have

$$\frac{\partial v_1}{\partial t}(x,t) = {}^{k_c}\!/_2\left(f'(x+k_c t) \ominus f'(x-k_c t)\right) + {}^1\!/_2\left(g(x+k_c t)+g(x-k_c t)\right).$$

Hence, u_1 is (i)-partial differentiable w.r.t. t and we obtain

$$\frac{\partial u_1}{\partial t}(x,t) = \frac{\partial v_1}{\partial t}(x,t) + \frac{\partial w}{\partial t}(x,t).$$

In what follows, we shall restrict our attention to the derivative of $\frac{\partial u_1}{\partial t}$ w.r.t. t. With the aid of Lemma 3.6, we deduce that $f'(x+k_c t) \ominus f'(x-k_c t)$ is (i)-partial differentiable w.r.t. t. It follows from Lemma 3.5 the function $g(x+k_c t)+g(x-k_c t)$ is (i)-partial differentiable w.r.t. t. It means that, the function $\frac{\partial v_1}{\partial t}$ is (i)-partial differentiable w.r.t. t and we derive

$$\frac{\partial^2 v_1}{\partial t^2}(x,t) = {}^{k_c^2}\!/_2\left(f''(x+k_c t)+f''(x-k_c t)\right) + {}^{k_c}\!/_2\left(g'(x+k_c t) \ominus g'(x-k_c t)\right).$$

From Lemma 2.11, we conclude that $\frac{\partial u_1}{\partial t}$ is (i)-partial differentiable w.r.t. t and

$$\frac{\partial^2 u_1}{\partial t^2}(x,t) = \frac{\partial^2 v_1}{\partial t^2}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t).$$

Finally, we show that u_1 satisfies Problem (3.1) for all $(x,t) \in R \times (0,+\infty)$. Since the real function $f_c^{"} > 0$,

Therefore

$\frac{\partial^2 u_c}{\partial x^2}(x,t) = {}^1/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right) > 0$

and we deduce

$k\odot\frac{\partial^2 u_1}{\partial x^2}(x,t) = (k_l, k_c^2, k_r)\odot({}^1/_2\left(f_l^{"}(x+k_c t) + f_l^{"}(x-k_c t)\right) + {}^1/_{2k_c}\left(g_l'(x+k_c t) - g_l'(x-k_c t)\right)$
$+\frac{k_l-k_c^2}{4k_c}t\left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right), {}^1/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right), {}^1/_2\left(f_r^{"}(x+k_c t) + f_r^{"}(x-k_c t)\right) +$
${}^1/_{2k_c}\left(g_r'(x+k_c t) - g_r'(x-k_c t)\right) + \frac{k_r-k_c^2}{4k_c}t\left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right)$
$= ({}^{k_l}/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right) + k_c^2({}^1/_2\left(f_l^{"}(x+k_c t) + f_l^{"}(x-k_c t)\right) + {}^1/_{2k_c}\left(g_l'(x+k_c t) - g_l'(x-k_c t)\right)$
$+\frac{k_l-k_c^2}{4k_c}t\left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right) - {}^{k_c^2}/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right), {}^{k_c^2}/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right)$
$,\frac{k_r}{2}\left(f_c^{"}(x+k_c t) - f_c^{"}(x-k_c t)\right) + k_c^2({}^1/_2\left(f_r^{"}(x+k_c t) + f_r^{"}(x-k_c t)\right) + {}^1/_{2k_c}\left(g_r'(x+k_c t) - g_r'(x-k_c t)\right) +$
$+\frac{k_r-k_c^2}{4k_c}t\left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right) - {}^{k_c^2}/_2\left(f_c^{"}(x+k_c t) + f_c^{"}(x-k_c t)\right)) = \frac{\partial^2 u_1}{\partial t^2}(x,t).$

It is easy to check that $u_1(x,0) = f(x)$, $\frac{\partial u_1}{\partial t}(x,0) = g(x)$.

Then, $u_1(x,t)$ is a triangular fuzzy function. As a result, u_1 is a solution for Problem (3.1).

Case 2. Similar to the proof of Case 1, we observe the fuzzy function w is (i)-partial *differentiable* w.r.t. *x*,*t* and the

functions $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}$ are (i)-partial differentiable w.r.t. *x*,*t* respectively.

On the other hand, employing Lemma 3.5, we deduce that $f(x+k_c t) + f(x-k_c t)$ is (ii)-partial differentiable w.r.t.

x. By recalling Lemma 3.8, the function $\int_{x-k_c t}^{x+k_c t} g(s)ds$ is (i)-partial differentiable w.r.t. x. Then from Lemma 2.11,

we conclude that v_2 is (ii)-partial differentiable w.r.t. *x* and we have

$\frac{\partial v_2}{\partial x}(x,t) = {}^1/_2\left(f'(x+k_c t) + f'(x-k_c t)\right) + {}^1/_{2k_c}\left(g(x+k_c t) \ominus g(x-k_c t)\right).$

By recalling Lemma 2.11, we deduce that u_2 is (ii)-partial differentiable w.r.t. *x* and

$\frac{\partial u_2}{\partial x}(x,t) = \frac{\partial v_2}{\partial x}(x,t) + (-1)((-1)\frac{\partial w}{\partial x}(x,t)).$

It follows from Lemma 3.5 that $f'(x+k_c t) + f'(x-k_c t)$ is (i)-partial differentiable w.r.t. x. From Lemma 3.6 we conclude that $g(x+k_c t) \ominus g(x-k_c t)$ is (i)-partial differentiable w.r.t. x. Hence, from Lemma 2.11, we conclude that $\frac{\partial v_2}{\partial x}$ is (i)-partial differentiable w.r.t. *x* and

$\frac{\partial^2 v_2}{\partial x^2}(x,t) = {}^1/_2\left(f"(x+k_c t) + f"(x-k_c t)\right) + {}^1/_{2k_c}\left(g'(x+k_c t) \ominus g'(x-k_c t)\right).$

Therefore, it follows from Lemma 2.11, that $\frac{\partial u_2}{\partial x}$ is (i)-partial differentiable w.r.t. *x* and

$\frac{\partial^2 u_2}{\partial x^2}(x,t) = \frac{\partial^2 v_2}{\partial x^2}(x,t) + \frac{\partial^2 w}{\partial x^2}(x,t).$

From Lemma 3.5 we deduce that $f(x+k_c t) + f(x-k_c t)$ is (ii)-partial differentiable w.r.t. t. It follows from Lemma

3.8 that the function $\int_{x-k_c t}^{x+k_c t} g(s)ds$ is (i)-partial differentiable w.r.t. t. Therefore, from Lemma 2.11, v_2 is (ii)-partial
differentiable w.r.t. *t* and we deduce

$$\frac{\partial v_2}{\partial t}(x,t) = {}^{k_c}/_2 \left(f'(x+k_c t) \ominus f'(x-k_c t)\right) + {}^1/_2 \left(g(x+k_c t) + g(x-k_c t)\right).$$

Hence from Lemma 2.11, u_2 is (ii)-partial differentiable w.r.t. t and

$$\frac{\partial u_2}{\partial t}(x,t) = \frac{\partial v_2}{\partial t}(x,t) + (-1)((-1)\frac{\partial w}{\partial t}(x,t)).$$

By recalling Lemma 3.6, we conclude that $f'(x+k_c t) + f'(x-k_c t)$ is (i)-partial differentiable w.r.t. t. From Lemma 3.5 we conclude that $g(x+k_c t) + g(x-k_c t)$ is (i)-partial differentiable w.r.t. t. It follows from Lemma 2.11 that the function is (i)-partial differentiable w.r.t. *t* and

$$\frac{\partial^2 v_2}{\partial t^2}(x,t) = {}^{k_c^2}/_2 \left(f''(x+k_c t) + f''(x-k_c t)\right) + {}^{k_c}/_2 \left(g'(x+k_c t) \ominus g'(x-k_c t)\right).$$

From Lemma 2.11, that $\frac{\partial u_2}{\partial t}$ is (i)-partial differentiable w.r.t. *t* and

$$\frac{\partial^2 u_2}{\partial t^2}(x,t) = \frac{\partial^2 v_2}{\partial t^2}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t).$$

Finally, we show that u_2 satisfies Problem (3.1) for all $(x,t) \in R \times (0,+\infty)$. Since the real function $f_c'' > 0$,
Therefore

$$\frac{\partial^2 u_c}{\partial x^2}(x,t) = {}^1/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right) > 0,$$

and we deduce

$$k \odot \frac{\partial^2 u_2}{\partial x^2}(x,t) = (k_l, k_c^2, k_r) \odot ({}^1/_2 \left(f_r''(x+k_c t) + f_r''(x-k_c t)\right) + {}^1/_{2k_c} \left(g_l'(x+k_c t) - g_l'(x-k_c t)\right)$$
$$+ \frac{k_l - k_c^2}{4k_c} t \left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right), {}^1/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right), {}^1/_2 \left(f_l''(x+k_c t) + f_l''(x-k_c t)\right) +$$
$${}^1/_{2k_c} \left(g_r'(x+k_c t) - g_r'(x-k_c t)\right) + \frac{k_r - k_c^2}{4k_c} t \left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right)$$
$$= ({}^{k_l}/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right) + k_c^2({}^1/_2 \left(f_r''(x+k_c t) + f_r''(x-k_c t)\right) + {}^1/_{2k_c} \left(g_l'(x+k_c t) - g_l'(x-k_c t)\right)$$
$$+ \frac{k_l - k_c^2}{4k_c} t \left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right) - {}^{k_c^2}/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right), {}^{k_c^2}/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right)$$
$$, \frac{k_r}{2} \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right) + k_c^2({}^1/_2 \left(f_l''(x+k_c t) + f_l''(x-k_c t)\right) + {}^1/_{2k_c} \left(g_r'(x+k_c t) - g_r'(x-k_c t)\right) +$$
$$+ \frac{k_r - k_c^2}{4k_c} t \left(f_c^{(3)}(x+k_c t) - f_c^{(3)}(x-k_c t)\right) - {}^{k_c^2}/_2 \left(f_c''(x+k_c t) + f_c''(x-k_c t)\right)) = \frac{\partial^2 u_2}{\partial t^2}(x,t).$$

It is easy to check that $u_2(x,0) = f(x)$, , $\frac{\partial u_2}{\partial t}(x,0) = g(x)$.

Consequently, u_2 is a solution for Problem (3.1). This completes the proof. □

Example 3.9 Consider the following fuzzy wave equation

$$u_{tt} = (0,4,8) \odot u_{xx} \qquad (3.2)$$
$$u(x, 0) = (2, 4, 6)\, e^x \quad x \in R,$$
$$u_t(x,0) = (-4,0,4)e^x \quad x \in R.$$

Then we set k = (0, 4, 8), f (x) = (2, 4, 6)e^x *and* g(x)=$(-4, 0,4)e^x$. *Employing Lemma 2.15 , we* deduce f, f' are (i)-differentiable for all $x \in$ R. In a similar way, we observe the functions g and g' are (i)-differentiable for all $x \in$ R. From Case 1 of Theorem 3.2, u_1 is a solution of Problem (3.2).

We compute it as follows

$u_1(x,t) = (1,2,3)e^{x+2t} + (1,2,3)e^{x-2t} + \int_{x-2t}^{x+2t}(-1,0,1)e^s ds + (-2,0,2)\int_0^t \int_{x-2(t-s)}^{x+2(t-s)}(e^{y+2s} + e^{y-2s})\, dyds$
$= \left(e^{x+2t} + e^{x-2t}, 2(e^{x+2t} + e^{x-2t}), 3(e^{x+2t} + e^{x-2t})\right) + (-e^{x+2t} + e^{x-2t}, 0, e^{x+2t} - e^{x-2t})$
$+(-2,0,2)\int_0^t((e^{x+2t-2s+2s} + e^{x+2t-2s-2s}) - (e^{x-2t+2s+2s} + e^{x-2t+2s-2s}))ds$
$= \left(2e^{x-2t} - 2t(e^{x+2t} - e^{x-2t}), 2(e^{x+2t} + e^{x-2t}), 4e^{x+2t} + 2e^{x-2t} + 2t(e^{x+2t} - e^{x-2t})\right).$

Moreover, we have

$(\underline{u}_1)_0 = (2+2t)e^{x-2t} - 2te^{x+2t},$
$(\overline{u}_1)_0 = (4+2t)e^{x+2t} + (2-2t)e^{x-2t}$

Graphical representation of $(\underline{u}_1)_0, (\overline{u}_1)_0$ have been in Figure 1.

4. Comparison Between Uncertainty of Solution of Wave Equations Under Different Conditions

Our purpose in this section is to compare the uncertainty of the solutions of Problem (3.1) under the influence of the uncertainty of the speed and the initial values. To this end, in the following example we consider Problem (3.1) with different conditions and we compare the uncertainty of their solutions together. As a measure of the uncertainty we use the length of the 0-level set. So, if we say increasing uncertainty. We understand increasing length of the 0-level set.

Example 4.1 Consider seven problems as follows

$u_{tt} = 4u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = 4e^x \quad x \in R, \quad (4.1)$
$u_t(x,0) = 0 \quad x \in R.$

Where k, f, g are real.

$u_{tt} = 4u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = (2,4,6)e^x \quad x \in R, \quad (4.2)$
$u_t(x,0) = 0 \quad x \in R.$

Where k, g are real and f is fuzzy.

$u_{tt} = 4u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = 4e^x \quad x \in R, \quad (4.3)$
$u_t(x,0) = (-4,0,4)e^x \quad x \in R.$

Where k, f are real and g is fuzzy.

$u_{tt} = 4u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = (2,4,6)e^x \quad x \in R, \quad (4.4)$
$u_t(x,0) = (-4,0,4)e^x \quad x \in R.$

Where k is real and f, g are fuzzy.

$u_{tt} = (0,4,8) \odot u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = 4e^x \quad x \in R, \quad (4.5)$
$u_t(x,0) = 0 \quad x \in R.$

Where f, g are real and k is fuzzy.

$u_{tt} = (0,4,8) \odot u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = (2,4,6)e^x \quad x \in R, \quad (4.6)$
$u_t(x,0) = 0 \quad x \in R.$

Where g is real and k, f are fuzzy.

$u_{tt} = (0,4,8) \odot u_{xx} \quad (x,t) \in R \times (0,+\infty)$
$u(x,0) = 4e^x \quad x \in R, \quad (4.7)$
$u_t(x,0) = (-4,0,4) \quad x \in R.$

Where f is real and k, g are fuzzy.

The solution of Problem (4.1) is

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$u(x,t) = {}^1\!/_2\,(4e^{x+2t} + 4e^{x-2t})$

For $\alpha = 0$, we have

$\underline{u}_0 = \overline{u}_0 = {}^1\!/_2\,(4e^{x+2t} + 4e^{x-2t})$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 2.

We call Theorem 4.2 from [1] and obtain the solution of Problem (4.2). Therefore, we have

$$u(x,t) = {}^1\!/_2\left((2,4,6)e^{x+2t} + (2,4,6)e^{x-2t}\right) = \left(e^{x+2t} + e^{x-2t}, 2(e^{x+2t} + e^{x-2t}), 3(e^{x+2t} + e^{x-2t})\right),$$

We have

$\underline{u}_0 = e^{x+2t} + e^{x-2t}, \overline{u}_0 = 3(e^{x+2t} + e^{x-2t})$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 3.

Applying Theorem 4.2 from [1] and obtain the solution of Problem (4.3). Therefore, we have

$$u(x,t) = {}^1\!/_2\left((4,4,4)e^{x+2t} + (4,4,4)e^{x-2t}\right) + {}^1\!/_4\int_{x-2t}^{x+2t}(-4,0,4)e^s ds = (e^{x+2t} + 3e^{x-2t}, 2(e^{x+2t} + e^{x-2t}), 3e^{x+2t} + e^{x-2t}).$$

We have,

$\underline{u}_0 = e^{x+2t} + 3e^{x-2t}, \overline{u}_0 = 3e^{x+2t} + e^{x-2t}$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 4.

Applying Theorem 4.2 from [1] and obtain the solution of Problem (4.4). Therefore, we have

$$u(x,t) = {}^1\!/_2\left((2,4,6)e^{x+2t} + (2,4,6)e^{x-2t}\right) + {}^1\!/_4\int_{x-2t}^{x+2t}(-4,0,4)e^s ds = (2e^{x-2t}, 2(e^{x+2t} + e^{x-2t}), 4e^{x+2t} + 2e^{x-2t}).$$

We have,

$\underline{u}_0 = 2e^{x-2t}, \overline{u}_0 = 4e^{x+2t} + 2e^{x-2t}$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 5.

Applying the method of previous section, we obtain the solution of Problem (4.5). Therefore, we have

$$u(x,t) = \left(2(e^{x+2t} + e^{x-2t}) - 2t(e^{x+2t} - e^{x-2t}), 2(e^{x+2t} + e^{x-2t}), 2(e^{x+2t} + e^{x-2t}) + 2t(e^{x+2t} - e^{x-2t})\right).$$

We have,

$\underline{u}_0 = 2(e^{x+2t} + e^{x-2t}) - 2t(e^{x+2t} - e^{x-2t}), \overline{u}_0 = 2(e^{x+2t} + e^{x-2t}) + 2t(e^{x+2t} - e^{x-2t})$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 6.

Applying the method of previous section, we obtain the solution of Problem (4.6) as

$$u(x,t) = {}^1\!/_2\,((2,4,6)e^{x+2t} + (2,4,6)e^{x-2t}) + (-1,0,1)\int_0^t\int_{x-2(t-s)}^{x+2(t-s)}(e^{y+2s} + e^{y-2s})\,dyds$$
$$= ((e^{x+2t} + e^{x-2t}) - 2t(e^{x+2t} - e^{x-2t}), 2(e^{x+2t} + e^{x-2t}), 3(e^{x+2t} + e^{x-2t}) + 2t(e^{x+2t} - e^{x-2t}).$$

We have

$\underline{u}_0 = (e^{x+2t} + e^{x-2t}) - 2t(e^{x+2t} - e^{x-2t}), \overline{u}_0 = 3(e^{x+2t} + e^{x-2t}) + 2t(e^{x+2t} - e^{x-2t})$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 7.

The solution of Problem (4.7) is

$$u(x,t) = {}^1\!/_2\,(4,4,4)(e^{x+2t} + e^{x-2t}) + {}^1\!/_4\int_{x-2t}^{x+2t}(-4,0,4)e^s ds + (-2,0,2)\int_0^t\int_{x-2(t-s)}^{x+2(t-s)}(e^{y+2s} + e^{y-2s})\,dyds$$
$$= (e^{x+2t} + 3e^{x-2t} - 2t(e^{x+2t} - e^{x-2t}), 2(e^{x+2t} + e^{x-2t}), 3e^{x+2t} + e^{x-2t} + 2t(e^{x+2t} - e^{x-2t})).$$

We have

$$\underline{u}_0 = e^{x+2t} + 3e^{x-2t} - 2t(e^{x+2t} - e^{x-2t}), \overline{u}_0 = 3e^{x+2t} + e^{x-2t} + 2t(e^{x+2t} - e^{x-2t})$$

Graphical representation of $\underline{u}_0, \overline{u}_0$ have been shown in Figure 8.

Now, let's compare these problems in two aspects. First, we compare the measure of the uncertainty of solution of problems that have the same speed and different initial values.

Figure 1 shows that $len([u(x,t)]_0)$ of Problem (3.2) is greater than that of (4.5), (4.6) and (4.7) for all (x,t). It means that the solution of Problem (3.2) is fuzzier than that of (4.5), (4.6) and (4.7). Therefore, Problem (3.2) is fuzzier than (4.5), (4.6) and (4.7). In a similar way, we conclude that the uncertainty of the solution of Problem (4.4) is greater than of (4.1), (4.2) and (4.3). As we can see, the uncertainty of the solution is influenced by the uncertainty of initial values. Secondly, we compare the measure of the uncertainty of solution of problems that have the different speed and same initial values.

We focus on Figures 1 and 5. We see that the solution of Problem (3.2) is fuzzier than that of (4.4). Moreover, Figures 2, 3, 4, 6, 7 and 8 show that the solution of Problem (4.5) is fuzzier than that of (4.1) and Problem (4.6) is fuzzier than that of (4.2) and Problem (4.7) is fuzzier than that of (4.3). As we can see, the uncertainty of the solution is influenced by the uncertainty of speed parameter. Finally, we can see that effect of uncertainty of speed on the solutions is more than the uncertainty of initial values.

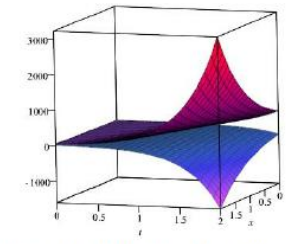

Figure 1. Solution of Problem (3.2)

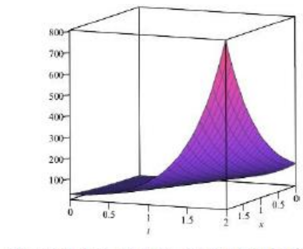

Figure 2. Solution of Problem (4.1)

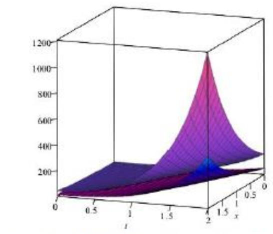

Figure 3. Solution of Problem (4.2)

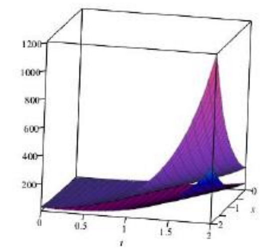

Figure 4. Solution of Problem (4.3)

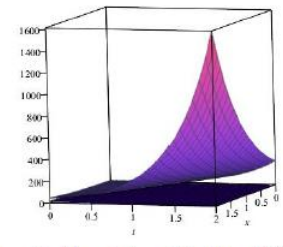

Figure 5. Solution of Problem (4.4)

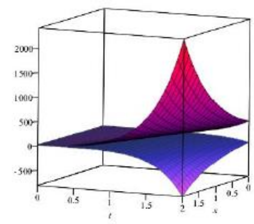

Figure 6. Solution of Problem (4.5)

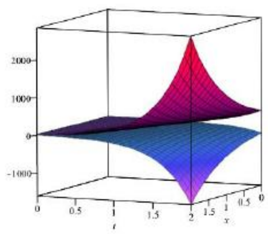

Figure 7. Solution of Problem (4.6)

References

1. R. Alikhani, F. Bahrami, S. Parvizi, Differential calculus of fuzzy multi-variable functions and its applications to fuzzy partial differential equations, Fuzzy sets and systems 375 (2019) 100-120.
2. R. Alikhani, F. Bahrami, Fuzzy partial differential equations under the cross product of fuzzy numbers, Information Sciences 494 (2019) 80-99.
3. T. Allahviranloo, Difference methods for fuzzy partial differential equations, Computational Methods in Applied Mathematics Comput. Methods Appl. Math 2 (3) (2006) 233 - 242.
4. T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, On fuzzy solutions for heat equation based on generalized Hukuhara differentiability, Fuzzy Sets Syst 265 (2015) 1-23.
5. T. Allahviranloo, M. Shafiee, Y. Nejatbakhsh, A note on fuzzy differential equations and the extension principle, Inf. Sci 179 (2009) 2049-2051.
6. S. Arshad, V. Lupulescu, Fractional differential equation with the fuzzy initial condition, Electronic Journal of Differential Equations 34 (2011) 1-8.
7. F. Bahrami, R. Alikhani, A. Khastan, Transport equation with fuzzy data, Iranian Journal of Fuzzy Systems 15 (2018) 67-78.
8. A.I. Ban, B. Bede, Cross product 0f L-R fuzzy numbers and applications. Annals of the University of Oradea, Fasc. Math 9 (2003) 95-108.
9. L.C. Barros, L.T. Gomes, P.A. Tonelli, Fuzzy differential equations: an approach via fuzzification of the derivative operator, Fuzzy Sets Syst 230 (2013) 39-52.
10. B. Bede, Mathematics of fuzzy sets and fuzzy logic, Springer, London, 2013.
11. B. Bede, S.G. Gal, Almost periodic fuzzy-number-valued functions, Fuzzy Sets and Systems 147 (2004) 385-403.
12. B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (3) (2005) 581-599.
13. B. Bede, I.J. Rudas, A.L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences 177 (2007) 1648-1662.
14. A.M. Bertone, R.M. Jafelice, L.C. de Barros, R.C. Bassanezi, On fuzzy solutions for partial differential equations, Fuzzy Sets and Systems 219 (2013) 68-80.
15. J.J. Buckley, Th. Feuring, Introduction to fuzzy partial differential equations, Fuzzy Sets and Systems 105 (2) (1999) 241-248.
16. Y. Chalco-Cano, W.A. Lodwick, B. Bede, Fuzzy differential equations and Zadeh's extension principle, in 2011 Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS) 2011 1-5.
17. D. Dubios, H. Prade, Fuzzy numbers: An overview. In: Analysis of Fuzzy Information, Mathematical Logic 1 (1987) 3-39.
18. N.A. Gasilov, A.G. Fatullayev, S.E. Amrahov, A. Khastan, A new approach to fuzzy initial value problem, Soft Comput 18 (2014) 217-225.
19. Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy, A. Armand, A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, Fuzzy Sets and Systems 309 (2017) 81-97.
20. R.M. Jafelice, C.G. Almeida, J.F. Meyer, H.L. Vasconcelos, Fuzzy parameter in a partial differential equation model for population dispersal of leaf-cutting ants, Nonlinear Analysis: Real

World Applications 12 (2011) 3397-3412.

21. O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.
22. A. Khastan, R. Rodrguez-Lopez, On the solutions to first order linear fuzzy differential equations, Fuzzy Sets and Systems 295 (2016) 114-135.
23. H.V. Long, N.T.K. Son, H.T.T. Tam, The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability, Fuzzy Sets and Systems 309 (2017) 35-63.
24. J.J. Nieto, The Cauchy problem for continuous fuzzy differential equations, Fuzzy Sets Syst 102 (1999) 259-262.
25. M. Puri, D. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91 (2) (1983) 552-558.
26. S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330.

Authors

Author's picture should be in grayscale.

Picture size should be absolute 3.18cm in height and absolute 2.65cm in width

Sanam. Parvizi