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Three Interesting Sequences

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Abstract

In mathematics, sequences play an important role in understanding of patterns and structures of mathematical objects. Among several amusing mathematical sequences, I discuss about three sequences in this paper. The subtle connections arising between them have been discussed in detail in this paper. Finally, the limiting behaviors of all these three sequences are found to be equal, proving asymptotically they behave in same way.

***Keywords**: Jacobsthal Sequence, Jacobsthal – Lucas Sequence, Mersenne Sequence, Recursive Relation, Binet's Formula, Asymptotic Behavior*

1. Introduction

In this paper, we wish to define numbers defined through three sequences namely, Jacobsthal numbers, Jacobsthal – Lucas numbers, Mersenne numbers. Sequences arise in practical problems and had been studied by the corresponding author (see **[1]**, **[2], [3], [4]**) and by various mathematicians whose applications arise in several fields of Science and Engineering. This paper will explain some of the interesting connections between the numbers formed by aforementioned three sequences and discuss about the asymptotic behavior of ratio of successive terms.

2. Definition

2.1 Jacobsthal Sequence

The sequence $\left\{J_n\right\}_{n=0}^{\infty}$ defined by the recursive relation $J_{n+2} = J_{n+1} + 2J_n,\ J_0 = 0,\ J_1 = 1 \quad (2.1)$ is called the Jacobsthal sequence. The numbers generated through (2.1) given by 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, . . . are called Jacobsthal numbers.

2.2 Jacobsthal – Lucas Sequence

The sequence $\left\{j_n\right\}_{n=0}^{\infty}$ defined by the recursive relation $j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2,\ j_1 = 1 \quad (2.2)$ is called the Jacobsthal – Lucas sequence. The numbers generated through (2.2) given by 2, 1, 5, 7, 17, 31, 65, 127, 257, 511, . .. are called Jacobsthal – Lucas numbers.

2.3 Mersenne Sequence

The sequence $\left\{M_n\right\}_{n=0}^{\infty}$ defined by the recursive relation $M_{n+1} = 2M_n + 1,\ M_0 = 0, M_1 = 1 \quad (2.3)$ is called the Mersenne sequence. The numbers generated through (2.3) given by 0, 1, 3, 7, 15, 31, 63, 127, 255, 511, . . . are called Mersenne numbers named after French mathematical monk Marin Mersenne.

Having defined the three sequences through equations (2.1), (2.2) and (2.3), we first try to establish the general recursion formula (Binet's Formula) defined by those equations.

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3. Deriving Binet's Formulas

Using the shift operator, defined by $E^r x_n = x_{n+r}$ the characteristic equation defined through equations (2.1) and (2.2) is $m^2 - m - 2 = 0$. This equation has roots $m = -1, 2$. Hence the solution (*n*th term) of (2.1) as well as (2.2) will be of the form $x_n = \alpha(-1)^n + \beta 2^n$ (3.1)

For obtaining solution of (2.1), we use the initial conditions $x_0 = 0, x_1 = 1$ to get $J_n = \dfrac{2^n - (-1)^n}{3}$ (3.2)

For obtaining solution of (2.2), we use the initial conditions $x_0 = 2, x_1 = 1$ to get $j_n = 2^n + (-1)^n$ (3.3)

Now the characteristic equation of (2.3) is $m - 2 = 0$ whose solution is $m = 2$. The solution of the homogenous equation of (2.3) is given by $M_n = \alpha 2^n$. The particular solution of (2.3) is given by $\dfrac{1}{E-2}(1) = \dfrac{1}{E-2}(1)^n = \dfrac{1}{1-2}(1)^n = -(1)^n = -1$. Hence, the general solution of (2.3) is given by $M_n = \alpha 2^n - 1$ (3.4). Using the initial condition $M_0 = 0$ we get $M_n = 2^n - 1$ (3.5).

Equations (3.2), (3.3) and (3.5) provide the Binet's formula (General Term Formula) for the sequences defined in (2.1), (2.2) and (2.3) respectively.

4. Connection between the three Sequences

In view of (2.1) and (2.2) we see both Jacobsthal and Jacobsthal – Lucas sequences are identical having same recurrence relation, the only difference being their initial conditions. Hence we derive the relationship between Mersenne sequence with that of Jacobsthal and Jacobsthal – Lucas sequences. Using the respective Binet's Formula we also produce equations connecting all three sequences through the following theorems.

4.1 Theorem 1

If M_n, J_n, j_n are *n*th terms of Mersenne, Jacobsthal and Jacobsthal – Lucas sequences respectively, then

(a) For *n* even we have $3J_n = M_n$ (4.1), $j_n = M_n + 2$ (4.2)

(b) For *n* odd we have $3J_n = M_n + 2$ (4.3), $j_n = M_n$ (4.4)

(c) For all *n*, we have $3J_n + j_n = 2(M_n + 1)$ (4.5)

Proof: For proving this, we make use of the respective Binet's Formulas proved in section 3.

(a) If *n* is even then $(-1)^n = 1$ for all whole numbers *n*. Hence using (3.2),(3.3) and (3.5), we have $3J_n = 2^n - 1 = M_n$, $j_n = 2^n + 1 = (2^n - 1) + 2 = M_n + 2$ giving equations (4.1) and (4.2)

(b) If n is odd then $(-1)^n = -1$ for all whole numbers n. Hence from (3.2), (3.3) and (3.5), we have $3J_n = 2^n + 1 = (2^n - 1) + 2 = M_n + 2$, $j_n = 2^n - 1 = M_n$ giving equations (4.3) and (4.4)

(c) $3J_n + j_n = \left[2^n - (-1)^n\right] + \left[2^n + (-1)^n\right] = 2 \times 2^n = 2 \times (2^n - 1 + 1) = 2(M_n + 1)$ which is equation (4.5)

This completes the proof.

4.2 Theorem 2

Let $\{J_n\}_{n=0}^{\infty}, \{j_n\}_{n=0}^{\infty}, \{M_n\}_{n=0}^{\infty}$ are Jacobsthal, Jacobsthal – Lucas, Mersenne sequences.

(a) If n is even then $\lim \frac{M_n + J_n}{J_n} = 4$ (4.6), $\lim \frac{M_n + j_n + 2}{j_n} = 2$ (4.7) as $n \to \infty$

(b) If n is odd then $\lim \frac{M_n + J_n + 2}{J_n} = 4$ (4.8), $\lim \frac{M_n + j_n}{j_n} = 2$ (4.9) as $n \to \infty$

Proof: We prove this, we make use of equations (4.1) to (4.4) form Theorem 1.

(a) If n is even then from equations (4.1) and (4.2), we have

$$\lim \frac{M_n + J_n}{J_n} = \lim \frac{3J_n + J_n}{J_n} = 4, \lim \frac{M_n + j_n + 2}{j_n} = \lim \frac{(j_n - 2) + j_n + 2}{j_n} = 2 \text{ as } n \to \infty.$$

(b) If n is odd then from equations (4.3) and (4.4), we have

$$\lim \frac{M_n + J_n + 2}{J_n} = \lim \frac{(3J_n - 2) + J_n + 2}{J_n} = 4, \lim \frac{M_n + j_n}{j_n} = \lim \frac{j_n + j_n}{j_n} = 2 \text{ as } n \to \infty$$

This proves equations (4.6) to (4.9) as desired.

4.3 Theorem 3

Let $\{J_n\}_{n=0}^{\infty}, \{j_n\}_{n=0}^{\infty}, \{M_n\}_{n=0}^{\infty}$ are Jacobsthal, Jacobsthal – Lucas, Mersenne sequences.

(a) If n is even then $\lim \frac{M_n}{J_n} = 3$ as $n \to \infty$

(b) If n is odd then $\lim \frac{M_n}{j_n} = 1$ as $n \to \infty$

Proof:

(a) If n is even, then from (4.1), then $\lim \frac{M_n}{J_n} = \lim \frac{3J_n}{J_n} = 3$ as $n \to \infty$

(b) If n is odd, then from (4.4), then $\lim \frac{M_n}{j_n} = \lim \frac{j_n}{j_n} = 1$ as $n \to \infty$

This completes the proof.

5. Asymptotic behavior of three sequences

We now prove a well – known but useful theorem.

5.1 Theorem 4

If $|x| < 1$ i.e. $-1 < x < 1$ then the sequence $\{x^n\}_{n=0}^{\infty}$ converges to 0. That is, $x^n \to 0$ as $n \to \infty$.

Proof: First we observe that if $x = 0$ then $x^n = 0$ for all n, so the theorem is obvious for this choice. If $0 < x < 1$ then $0 < x^n < 1$. Also, the terms of the sequence $\{x^n\}_{n=0}^{\infty}$ are non-increasing, bounded below by its greatest lower bound 0. Hence, $x^n \to 0$ as $n \to \infty$.

If $-1 < x < 0$, first we consider the even subscript terms subsequence given by $\{1, x^2, x^4, x^6, ...\}$. We note that the even terms subsequence $\{1, x^2, x^4, x^6, ...\}$ is such that each term is positive, non-increasing and bounded below by 0. Also the greatest lower bound for the terms in this subsequence is 0. Hence, for n even, $x^n \to 0$ as $n \to \infty$.

Now for $-1 < x < 0$, we consider the odd subscript terms subsequence given by $\{x, x^3, x^5, x^7, ...\}$ Since $-1 < x < 0$ the odd terms subsequence $\{x, x^3, x^5, x^7, ...\}$ is such that each term is negative, non-decreasing and bounded above by 0. Also the least upper bound for the terms in this subsequence is 0. Hence, for n odd, $x^n \to 0$ as $n \to \infty$. Thus if $-1 < x < 0$ then both even and odd subscript subsequences converges to 0, proving that for all $-1 < x < 0$, $x^n \to 0$ as $n \to \infty$.

This completes the proof.

We now use Theorem 4 to derive the following important consequence of the three sequences discussed above.

5.2 Theorem 5

If $\{J_n\}_{n=0}^{\infty}, \{j_n\}_{n=0}^{\infty}, \{M_n\}_{n=0}^{\infty}$ are Jacobsthal, Jacobsthal – Lucas, Mersenne sequences then

$$(5.1)\ \lim \frac{J_{n+1}}{J_n} = \lim \frac{j_{n+1}}{j_n} = \lim \frac{M_{n+1}}{M_n} = 2 \quad \text{as } n \to \infty.$$

Proof: We use Binet's Formulas (3.2), (3.3) and (3.5) and Theorem 4 to prove this theorem. We see that as $n \to \infty$,

$$\lim \frac{J_{n+1}}{J_n} = \lim \frac{\left[\dfrac{2^{n+1}-(-1)^{n+1}}{3}\right]}{\left[\dfrac{2^n-(-1)^n}{3}\right]} = \lim \frac{\left[2+\left(\dfrac{-1}{2}\right)^n\right]}{\left[1-\left(\dfrac{-1}{2}\right)^n\right]} = \frac{2+0}{1-0} = 2$$

$$\lim \frac{j_{n+1}}{j_n} = \lim \frac{2^{n+1}+(-1)^{n+1}}{2^n+(-1)^n} = \lim \frac{\left[2-\left(\dfrac{-1}{2}\right)^n\right]}{\left[1+\left(\dfrac{-1}{2}\right)^n\right]} = \frac{2-0}{1+0} = 2$$

$$\lim \frac{M_{n+1}}{M_n} = \lim \frac{2^{n+1}-1}{2^n-1} = \lim \frac{\left[2-\left(\dfrac{1}{2}\right)^n\right]}{\left[1-\left(\dfrac{1}{2}\right)^n\right]} = \frac{2-0}{1-0} = 2$$

Thus, we obtain equations in (5.1).

This completes the proof.

6. Conclusion

In this paper, through Theorem 1, equations (4.1) to (4.5) provide the relationship between the terms of the three sequences. Theorems 2 and 3 provide the ratios of the terms of Mersenne sequence as compared to Jacobsthal and Jacobsthal – Lucas sequences for even and odd subscripts. In Theorem 5, equation (5.1) provide the asymptotic behavior of the three sequences conveying the fact that the (n+1)th term of all the three sequences is roughly twice that of its nth term if n is very large. Thus through simple definitions, we found the inter-relationship between the terms of the three sequences and proved that in the limiting case the ratio of the successive terms is always 2.

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