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N – Policy Retrial Queueing System With Bernoulli Working Vacations

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ABSTRACT

We consider M/G/1 Retrial Queueing system with Bernoulli Working Vacation under N-policy. The detailed descriptions of the model are discussed. We develop the steady state differential equation for the retrial queueing system. The probability generating function for the numbers of customers in the orbit as well as the system when the server is idle, busy and on working vacation are found using supplementary variable technique. We derive some system state probabilities, mean number of customers in the system and its orbit of this model. We analyze some special cases for major results; a brief conclusion and summary of their project are presented.

Keywords: Retrial queue, working vacation, N-Policy, Bernoulli schedule

1.INTRODUCTION

Theoretical research into the properties of queues is started by A.K. Erlang in 1903. The Theory was further developed by Mollins in 1927 and then by Thomson Fry. A systematic approach to the problem was made byKendall in 1951 by using model terminology and since then significant work has been done in this directionIn past years, retrial queues with two types of customers have been widely studied by researchers Artalejo[1]. Gomaz-Corral[2] discussed about stochastic analysis of a single server retrial queue with retrial times. The server takes a vacation only when there are no customers in the system. Zhang et al. [3] discussed about M/G/1 queue with working vacation.In 2014, Arivudainambi et al. [4] developed the performance analysis of a single server retrial queue with working vacation and Gao et al. [5] examined M/G/1 retrial queue with general retrial times, working vacation and vacation interruption. The server enters the N-Policy state until N customers accumulated. Hur et al. [6] analyzed the M/G/1 system with N and T policy. Recently in 2018, Rajadurai et al. [7] developed the study on M/G/1 retrial G-queue with unreliable server under variant working vacations policy and vacation interruption. A working vacation queue with priority customer and vacation interruption was developed by Goswami and Selvaraju[8].

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2. DESCRIPTION OF THE MODEL

We investigate an M/G/1/N retrial queue with Bernoulli working vacation. The basic assumption of this model is described as follows:

2.1THE ARRIVAL PROCESS:

The Customer arrives into the system according to a Poisson process of rate λ.

2.2 THE RETRIAL PROCESS:

We assume that there is no waiting space and therefore an arriving customer finds the server free; the customer begins its service immediately.

2.3 THE REGULAR SERVICE PROCESS:

Whenever a new ordinary customer or retry customer arrives at the server idle state then the server immediately starts normal service for the arrivals.

2.4 WORKING VACATION UNDER N-POLICY

When the orbit is empty, the server goes to vacation state. At the time, when any arrival arrives, the server gives service to the customer in working vacation at lower rate. After the completion of the service, N-customer arrive, the server either idle to serve a new customer with probability *p*(single WV) in regular mode or leaves for another WV with probability *q*=1-*p* (multiple WV). When a vacation end and if there are *N*-customers in the orbit then the server switches to the normal working level(normal busy period).

3. STEADYSTATE ANALYSIS OF THE SYSTEM

For an M/G/1/N retrial queue with Bernoulli working vacation, we constructed the steady statedifference-differential equations based on the supplementary variable technique (SVT).

Let us define some notations in the following:

$N(t) \equiv$ The number of customers in the orbit at time *t*,

$\mathrm{X}(t) \equiv$ The state of the server at time *t*,

Where,

$$\mathrm{X}(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is in retrial,} \\ 2, & \text{if the server is in normal busy,} \\ 3, & \text{if the server is in working vacation} \end{cases}$$

In addition, let $R^0(t), S_b^0(t)$ and $S_v^0(t)$ be the elapsed times for retrial, normal busy and working vacation, respectively, at time t,

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In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_b(0)=0, S_b(\infty)=1$ $S_v(0)=0, S_v(\infty)=1$ are continuous at x=0.Then the functions $a(x), \mu_b(x)$ and $\mu_v(x)$ are the hazard rates for retrial, service and working vacation respectively.

$$i.e., a(x)dx = \frac{dR(x)}{1-R(x)}; \mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)}; \mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)}$$

Thus the supplementary variables are introduced in order to obtain a bivariate Markov process $\{C(t), N(t);\ t \geq 0\}$.

If C(t)=0 ,the server is free.

If $C(t)$ = 1 and $N(t) > 0$, then $R^0(t)$ represent the elapsed retrial time.

If $C(t)$ = 2 and $N(t) \geq 0$ then $S_b^0(t)$ corresponding to the elapsed time of the customer being served in normal busy period.

If C(t) = 3 and $N(t) \geq 0$ then $S_v^0(t)$ corresponding to the elapsed time of the customer being served in working vacation period.

Let $\{t_n;\ n = 1, 2, 3,...\}$ be the sequence of epochs at which either a normal service for ordinary or a lower service completion occurs.The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms an embedded Markov chain the retrial queueing system. If the embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda+\delta)$, where,

$$\rho = \left(R^*(\lambda+\delta) + \lambda R_*(\lambda+\delta)\right)\left(A_b' S_b^{*'}(\alpha) + S'(1)\right) + \delta\lambda\overline{R^*}(\lambda+\delta)\beta_p^{(1)}$$

The Steady state probabilities and limiting probability densities are defined as follows:

$$P_0(t) = P\{C(t)=0, N(t)=0\}$$

$$P_n(x)dx = \lim_{t\to\infty} P\{C(t)=1, N(t)=n, x < a(t) \leq x+dx\}, \text{for } t \geq 0, x \geq 0 \text{ and } n \geq 1$$

$$\pi_{b,n}(x)dx = \lim_{t\to\infty} P\{C(t)=2, N(t)=n, x \leq S_b^0(t) < x+dx\}, \text{for } t \geq 0, x \geq 0, n \geq 0$$

$$Q_{v,n}(x)dx = \lim_{t\to\infty} P\{C(t)=3, N(t)=n, x \leq S_v^0(t) < x+dx\}, \text{for } t \geq 0, x \geq 0, n \geq 0$$

The following probabilities are used in sequent sections:

- The probability that the system is empty at time t.
- The probability that at time t there are exactly n customers in the orbit with the elapsed normal service time of the test customer undergoing service lying in between x and $x+dx$.
- The probability that at time t there are exactly n customers in the orbit with the elapsed working time of the test customer undergoing service lying in between x and $x+dx$.

3.STEADY STATE EQUATIONS

The system of governing equations of server states as follows:

$$\lambda P_0 = \theta p Q_0 \tag{1}$$

$$(\lambda+\theta)Q_0=\theta qQ(z)+\int_0^\infty \pi_{b,0}(x)\mu_b(x)dx+\int_0^\infty Q_{v,0}(x)\mu_v(x)dx \quad (2)$$

$$\frac{dP_n(x)}{dx}+(\lambda+a(x))P_n(x)=0, n\geq 1 \quad (3)$$

$$\frac{d\pi_{b,0}(x)}{dx}+(\lambda+\mu_b(x))\pi_{b,0}(x)=0, n\geq 1 \quad (4)$$

$$\frac{d\pi_{b,n}(x)}{dx}+(\lambda+\mu_b(x))\pi_{b,n}(x)=\lambda\pi_{b,n-1}(x) \quad (5)$$

$$\frac{dQ_{v,0}(x)}{dx}+(\lambda+\theta+\mu_v(x))Q_{v,0}(x)=0, n=0 \quad (6)$$

$$\frac{dQ_{v,n}(x)}{dx}+(\lambda+\theta+\mu_v(x))Q_{v,n}(x)=\lambda Q_{v,n-1}(x), n\geq 1 \quad (7)$$

The steady state boundary conditions at $x = 0$ are

$$P_n(0)=\int_0^\infty \pi_{b,n}(x)\mu_b(x)dx+\int_0^\infty Q_{v,n}(x)\mu_v(x)dx, n\geq 1 \quad (8)$$

$$\pi_{b,0}(0)=\left(\int_0^\infty P_1(x)a(x)dx+\theta\int_0^\infty Q_{v,0}(x)dx+\lambda P_0\right), n=0 \quad (9)$$

$$\pi_{b,n}(0)=\left(\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\int_0^\infty P_n(x)dx+\theta\int_0^\infty Q_{v,n}(x)dx\right), n\geq 1 \quad (10)$$

$$Q_{v,n}(0)=\begin{cases}\lambda Q_n, 0\leq n\leq N-1\\ 0, n\geq N\end{cases} \quad (11)$$

The normalizing condition is

$$P_0+Q_0+\sum_{n=1}^\infty\int_0^\infty P_n(x)dx+\sum_{n=0}^\infty\left(\int_0^\infty \pi_{b,n}(x)dx+\int_0^\infty Q_{v,n}(x)dx\right)=1 \quad (12)$$

Computation of the steady state solution:

In the following, the probability generating function technique is used here to obtain the steadystate solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z|\leq 1$, as follows:

$$P(x,z)=\sum_{n=1}^\infty P_n(x)z^n;\ \pi_b(x,z)=\sum_{n=0}^\infty \pi_{b,n}(x)z^n;\ Q_v(x,z)=\sum_{n=0}^\infty Q_{v,n}(x)z^n;$$

Multiplying the steady state equation and steady state boundary condition (1) - (12) by z^n and summing over n, (n = 0,1,2...) and solving the partial differential equations, it follows that

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$$\frac{\partial P(x,z)}{\partial x}+(\lambda+a(x))P(x,z)=0 \tag{13}$$

$$\frac{\partial \pi_b(x,z)}{\partial x}+(\lambda(1-z)+\mu_b(x))\pi_b(x,z)=0 \tag{14}$$

$$\frac{\partial Q_v(x,z)}{\partial x}+(\theta+\mu_v(x)+\lambda b(1-z))Q_v(x,z)=0 \tag{15}$$

$$P(0,z)=\int_0^\infty \pi_b(x,z)\mu_b(x)dx+\int_0^\infty Q_v(x,z)\mu_v(x)dx-(\lambda+\theta p)Q(z) \tag{16}$$

$$\pi_b(0,z)=\left(\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\lambda\int_0^\infty P(x,z)dx+\lambda P_0+\theta\int_0^\infty Q_v(x,z)dx\right) \tag{17}$$

Solving the partial differential equations (13)-(15), it follows that

$$P(x,z)=P(0,z)\left[1-R(x)\right]e^{-\lambda x} \tag{18}$$

$$\pi_b(x,z)=\pi_b(0,z)\left[1-S_b(x)\right]e^{-A_b(z)x} \tag{19}$$

$$Q_v(x,z)=Q_v(0,z)\left[1-S_v(x)\right]e^{-A_v(z)x} \tag{20}$$

Where,

$A_p(z)=\lambda(1-z);\ A_b(z)=A_p(z);\ A_v(z)=(\theta+\lambda(1-z));$

Inserting Equation (19) in (18), we obtain

$$\pi_b(0,z)=\frac{P(0,z)}{z}\left[\left(R^*(\lambda)+\lambda zR^*(\lambda)\right)\right]+\lambda P_0+\lambda\theta Q(z)\overline{S_v^*}(\lambda(1-z)+\theta) \tag{21}$$

Using (18)-(21) in (16), we obtain

$$P(0,z)=\frac{1}{z}P(0,z)\begin{pmatrix}R^*(\lambda)+\lambda z\left(\frac{1-R^*(\lambda)}{\lambda}\right)+\lambda P_0+\theta Q_v(0,z)\left[\frac{1-S_v^*(\lambda(1-z)+\theta)}{\lambda(1-z)+\theta}\right]S_b^*(A_b(z))\\+\lambda Q(z)S_v^*(A_v(z))-(\lambda+\theta p)Q(z)\end{pmatrix} \tag{22}$$

Using equations (21)-(22) after making some manipulations, we obtain :

$$P(0,z)=\frac{zQ(z)\left[-\theta p\left(1-S_b^*(\lambda(1-z))\right)+\lambda\begin{pmatrix}\theta\left(\overline{S_v^*}(\lambda(1-z)+\theta)S_b^*(\lambda(1-z))\right)\\+S_v^*(\lambda(1-z)+\theta)-1\end{pmatrix}\right]}{z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*(A_b(z))} \tag{23}$$

$$\pi_b(0,z)=\frac{Q(z)\begin{Bmatrix}\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)\left[\lambda\left(S_v^*(\lambda(1-z)+\theta)\right)-\theta p\right]\\+\theta pz+\lambda z\theta\overline{S_v^*}(\lambda(1-z)+\theta)\end{Bmatrix}}{z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*(\lambda(1-z))} \tag{24}$$

Using equations (16) – (17) in (18) – (20), we obtain the limiting PGFs, The PGFs of the number of customers in the orbit when the server is retrial, busy and working vacation is given by:

$$P(z)=\frac{zQ(z)\overline{R^*}(\lambda)\left[-\theta p\left(1-S_b^*\left(\lambda(1-z)\right)\right)+\lambda\begin{pmatrix}\theta\left(\overline{S_v^*}\left(\lambda(1-z)+\theta\right)\right)S_b^*\left(\lambda(1-z)\right)\\+S_v^*\left(\lambda(1-z)+\theta\right)-1\end{pmatrix}\right]}{z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*\left(A_b(z)\right)} \tag{25}$$

$$\pi_b(z)=\frac{Q(z)\overline{S_b^*}\left(\lambda(1-z)\right)}{z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*\left(\lambda(1-z)\right)}\begin{bmatrix}\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)\left[\lambda\left(S_v^*\left(\lambda(1-z)+\theta\right)\right)-\theta p\right]\\+\theta pz+\lambda z\theta\overline{S_v^*}\left(\lambda(1-z)+\theta\right)\end{bmatrix} \tag{26}$$

$$Q_v(z)=\frac{\lambda Q(z)\left[1-S_v^*\left(\lambda(1-z)+\theta\right)\right]}{\lambda(1-z)+\theta} \tag{27}$$

The probability that the server is idle

$$Q(1)+P(1)+\pi_b(1)+Q_v(1)=1$$

$$Q(1)=\frac{R^*(\lambda)-\lambda E(S_b)}{R^*(\lambda)+E(S_b)(\theta p-\lambda)+\lambda\left(1-S_v^*(\theta)\right)\left(\frac{1}{\theta}+E(S_b)\right)}$$

Then the probability generating function $H(z)$ of the number of customer in the orbit is obtained by

$$H(z)=Q(z)+P(z)+\pi_b(z)+R(z)$$

$$H(z)=Q(z)\times\frac{\lambda(1-z)\begin{bmatrix}z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*\left(\lambda(1-z)\right)\left(1+\frac{\lambda}{\theta}V(z)\right)\\+z\left(1-R^*(\lambda)\right)(1-z)\left[\lambda\left(V(z)S_b^*\left(\lambda(1-z)\right)+S_v^*\left(\lambda(1-z)+\theta\right)-1\right)-\theta p\left(1-S_b^*\left(\lambda(1-z)\right)\right)\right]\\\left(1-S_b^*\left((1-z)\right)\right)\left[\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)\left(\lambda\left(S_v^*\left(\lambda(1-z)+\theta\right)-1\right)-\theta p+\theta pz+\lambda zV(z)\right)\right]\end{bmatrix}}{\lambda(1-z)\left[z-\left(R^*(\lambda)+\lambda z\overline{R}^*(\lambda)\right)S_b^*\left(\lambda(1-z)\right)\right]}$$

Where, $V(z)=\theta\overline{S_v^*}\left(\lambda(1-z)+\theta\right)$

Then the probability generating function $K(z)$ of the number of customer in the system is obtained by

$$K(z)=Q(z)+P(z)+z\pi_b(z)+zR(z)$$

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$$K(z)=Q(z)\times\frac{\lambda(1-z)\left[\begin{array}{l} z-\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)S_b^*(\lambda(1-z))\left(1+\lambda\left[\overline{S_v^*}(\lambda(1-z)+\theta)\right]\right) \\ +z\left(1-R^*(\lambda)\right)(1-z)\left[\lambda\left(V(z)S_b^*(\lambda(1-z))+S_v^*(\lambda(1-z)+\theta)-1\right)-\theta p\left(1-S_b^*(\lambda(1-z))\right)\right] \\ z\left(1-S_b^*((1-z))\right)\left[\left(R^*(\lambda)+\lambda z\overline{R^*}(\lambda)\right)\left(\lambda\left(S_v^*(\lambda(1-z)+\theta)-1\right)-\theta p+\theta pz+\lambda zV(z)\right)\right] \end{array}\right]}{\lambda(1-z)\left[z-\left(R^*(\lambda)+\lambda z\overline{R}^*(\lambda)\right)S_b^*(\lambda(1-z))\right]}$$

Where, $V(z)=\theta\overline{S_v^*}\left(\lambda(1-z)+\theta\right)$

4. PERFORMANCE MEASURES

4.1 System state probabilities

Let P be the steady state probability that the server is idle during the retrial time

$$P(1)=\frac{\lambda\overline{R^*}(\lambda)(Q(1))\left(\frac{\lambda}{\theta}\left(1-S_v^*(\theta)\right)\left(\frac{1}{\theta}+E(S_b)\right)+\theta pE(S_b)\right)}{R^*(\lambda)-\lambda\left(E(S_b)\right)}$$

The probability that the server is busy serving an ordinary customers is given by

$$\pi_b(1)=\frac{E(S_b)Q(1)\left[\lambda\left(1-S_v^*(\theta)\right)\left(R^*(\lambda)+\frac{\lambda}{\theta}\right)+\theta pR^*(\lambda)\right]}{R^*(\lambda)-\lambda E(S_b)}$$

The probability that the server is on working vacation is given by,

$$Q_v=Q_v(1)=\lambda Q(1)\left(\frac{1-S_v^*(\theta)}{\theta}\right)$$

4.2 Mean system size and orbit size

The expected number of customers in the system is obtained by differentiating with respect to z and evaluating at z=1:

$$L_s=K_s^{'}(1)=\lim_{z\to1}\frac{d}{dz}K_s(z)$$

The expected number of customers in the orbit is obtained by differentiating with respect to z and evaluating at z=1:

$$L_q=K_0^{'}(1)=\lim_{z\to1}\frac{d}{dz}K_0(z)$$

The average time a customer spends in the system and queue is expressed as:

$$W_s=\frac{L_s}{\lambda}\ \text{ and }\ W_q=\frac{L_q}{\lambda}$$

5. SPECIAL CASE

Case (i): No N-Policy

Put N=0; Our Model can be reduced to single server retrial queue with Bernoulli working vacation.

Case (ii): NoN – Policy, no multiple working vacation

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Put $\left(\theta, N\right)\rightarrow\left(0,0\right)$, Our Model can be reduced to M/G/1 Retrial queue with Single working vacation.

Case (iii): No N-Policy, No Bernoulli Working Vacations, No Retrial

Put $\left(\theta, N\right)\rightarrow\left(0,0\right)$, R*(λ) →1; Our Model can be reduced to M/G/1 queue.

6. CONCLUSION

In this work, we have studied an M/G/1/N Retrial Queue with Bernoulli Working Vacation. Because of wide applications, especially in computer and communication networks, manufacturing process and transportation, Bernoulli working vacation incorporated in the model makes our study morerealistic and more versatile in comparison to other existing models. The probability generating function for system size and orbit size are found by using the Supplementary variable Technique.System measures like the mean queue and system size are found.

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