Minimal and Maximal Solutions of Second-order Functional Difference Equations

S. Sindhuja[1], J. Daphy Louis Lovenia[2*], A.P. Lavanya[3],G.Gomathi Jawahar[4], D. Darling Jemima[5]

S. Sindhuja[1],Research Scholar, Department of Mathematics, Karunya Institute of Technologyand Sciences, Coimbatore, India, sindhusivaraj2211@gmail.com
J. Daphy Louis Lovenia[2*], Professor, Department of Mathematics, Karunya Institute of Technology and Sciences, Corresponding Author, Coimbatore, India,daphy@karunya.edu
A.P. Lavanya[3],Assistant Professor, Department of Mathematics, Sri Krishna college of Engineering and Technology, Coimbatore, India,algebralavanya@gmail.com
G.Gomathi Jawahar[4], Assistant Professor, Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore, India, gomathi@karunya.edu
D. Darling Jemima[5], Assistant Professor, Department of Computer Science and Engineering, Sri Ramakrishna Engineering college.jemima.d@srec.ac.in

Abstract

In this paper the minimal and maximal solutionsof second order functional difference equations with nonlinear boundary conditions of the form

$$\Delta^2 x(m_1-1)=\begin{Bmatrix}(F_1x)(m_1), m_1\in[1,W]=1,2,..W\\ f_2(x(0),x(W))=0,\end{Bmatrix}$$

where $(F_1x)(m_1)= f_1((k_1,x(m_1)),x(\beta_1(m_1)),x(\beta_2(m_1)),...x(\beta_r(m_1)))$ are analyzed.The above equation depends on r delayed arguments. We illustrate our results with numerical example.

Introduction

Let us consider the second-order functional difference equations

$$\Delta^2 x(m_1-1)=\begin{Bmatrix}(F_1x)(m_1), m_1\in[1,W]=1,2,..W\\ f_2(x(0),x(W))=0,\end{Bmatrix}(1)$$

and $(F_1x)(m_1)= f_1((m_1,x(m_1)),x(\beta_1(m_1)),x(\beta_2(m_1)),...x(\beta_r(m_1)))$ where $\Delta^2 x(m_1-1)=$ $x(m_1+1)-2x(m_1)+x(m_1-1)$

Assume

$A_1: f_1\in([0,W]\times\Re^{r+1},\Re),\ f_2\in N(\Re,\Re\times\Re),\beta_d\in N([1,W],[0,W])\beta_d(m_1)\leq m_1, d=1,2,...r,$

andanother type of equations:

$$\Delta^2 x(m_1) = \begin{Bmatrix} (F_1 x)(m_1), m_1 \in [1, W-1], \\ f_2(x(0), x(W)) = 0, \end{Bmatrix} \quad (2)$$

where $\Delta^2 x(m_1) = x(m_1+2) - 2x(m_1+1) + x(m_1)$

Assume

$$A_2 : f_1 \in ([0, W-1] \times \Re^{r+1}, \Re),\ f_2 \in N(\Re, \Re \times \Re), \beta_d \in N([0, W-1], [0, W-1])$$
$$\beta_d(m_1) \le m_1, d = 1,2,...r.$$

Difference equationsis a very fascinatingsubject because we can derive many complex properties based on simple formulation. Problems such as vibration of particles, phenomena in crystals, Statistics, electrical circuit analysis, dynamical systems, molecular chains, control theory etc. can be modeled by difference equations.

In the recent years, there has been an increasing interest in the study of second order functionaldifference equations with nonlinear boundary value conditions see [1-8].

Let $C = N([0,1], R)$ be a Banach space,a class of maps ω continuous on [0, W] with the norm

$$\|\omega\| = \max_{m_1 \in [o,W]} |\omega|.$$

A solution of (1) is $\omega \in C$ satisfies (1).

2. RESULTS

Theorem 1 Suppose
$\beta \in N([1,W],[0,W]), \beta(m_1) \le m_1, m_1 \in [0,W], x \in N([0,W], R), P, Q \in N([1,W], R_+), R_+ = [0,\infty)$
and

$$\begin{Bmatrix} \Delta^2 x(m_1 - 1) \le -(P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))x(m_1+1) - 2P(m_1)x(m_1) - Q(m_1)x(\beta(m_1)), m_1 \in [1,W] \\ x(0) \le 0. \end{Bmatrix}$$

Also, assume that

$$\delta_1 = \sum_{m_1=1}^{W} \sum_{d=1}^{T} Q(r) \prod_{d=1}^{m_1-1} [1 + P(d)] \le 1 \text{where} \prod_{d=1}^{0} ... = 1. \quad (3)$$

Then $x(m_1) \le 0, m_1 \in [0, W].$

Proof: Consider $x(m_1) > 0.$ Then there exists $m_0 \in (0, W]$ such that $x(m_0) > 0$.

Let $x(m_2) = \min_{m_1 \in [0, m_0]} x(m_1) = \mu \le 0$

Therefore,

$$[1+P(m_1)][1+P(m_1+1)]x(m_1+1)-2(1+p(m_1))x(m_1)+x(m_1-1)\le -\mu Q(m_1), m_1\in[1,m_0]$$

$$[1+P(m_1)][1+P(m_1+1)]x(m_1+1)S_{m_1-1}-2[1+P(m_1)]x(m_1)S_{m_1}+x(m_1-1)S_{m_1-1}\le -\mu Q(m_1)S_{m_1-1}, m_1\in[1,m_0],$$

$$S_0=1, S_{m_1}=\prod_{d=1}^{m_1}[1+P(d)], m_1\in[1,W].$$

Therefore,

$$\Delta^2[x(m_1-1)S_{m_1-1}]\le -\mu Q(m_1)S_{m_1-1}, m_1\in[1,m_0],$$

which implies that,

$$x(m_1+1)S_{m_1+1}-2x(m_1)S_{m_1}+x(m_1-1)S_{m_1-1}\le -\mu Q(m_1)S_{m_1-1}$$

Adding from m_2+1 to m_0

$$\Delta[x(m_0)S_{r_0}-\mu S_{m_2}]\le \sum_{d=m_2+1}^{m_0}-\mu Q(m_1)S_{m_1-1},$$

$$\Delta[-\mu S_{r_1}]\le -\mu[\sum_{r_1+1}^{r_0}Q(m_1)S_{m_1-1}]$$

Adding from r_1+1 to r_0

$$-\mu[S_{r_0+1}-\mu S_{r_1+1}]\le -\mu\sum_{r_1+1}^{r_0}\sum_{i=m_2+1}^{m_0}Q(m_1)S_{m_1-1}, \quad (4)$$

Hence

$$-\mu(S_r-S_{r_{1+2}})-x(r_{1+2})S_{r_{1+2}}\le -\mu\sum_{r_1+1}^{r_o}\sum_{d=m_2+1}^{m_0}Q(m_1)S_{m_1-1}$$

since S_{r_0} & S_{r_1+2} leads to zero $S_{m_1}\ge 1, \mu<0.$

$$-x(r_1+2)S_{r_1+2}\le \sum_{r_1+1}^{r_0}[\sum_{d=m_2+1}^{m_0}Q(m_1)S_{m_1-1}$$

ISSN: 2005-4238 IJAST

For $\mu < 0, S_{m_1} \geq 1$ is a contradiction.

Hence theproof.

Remark 1

Suppose
$\beta_d \in N([1,W],[0,W]), \beta_d(m_1) \leq m_1, m_1 \in [0,W], d \in [1,r], x \in N([0,W],R), P, Q_d \in N([1,W],R_+),$
$d \in [1,r]$
and

$$\begin{cases} \Delta^2 x(m_1) \leq -(P(m_1)(P(m_1+1)+P(m_1)+P(m_1+1))x(m_1+1) - 2P(m_1)x(m_1) - Q_d(m_1)x(\beta_d(m_1))), m_1 \in [1,W] \\ x(0) \leq 0. \end{cases}$$

Also, assume that

$$\delta_2 = \sum_{m_1=1}^{W}\sum_{d=1}^{T} Q(r)\prod_{d=1}^{m_1-1}[1+P(d)] \leq 1 \text{, where } Q(d) = \sum_{s=1}^{r} Q_s(d) \qquad (5)$$

Then $x(m_1) \leq 0. m_1 \in [0,W]$.

Remark 2 Consider the second order functional difference equation

$$\begin{cases} \Delta^2 x(m_1-1) = -(P(m_1)(P(m_1+1)+P(m_1)+P(m_1+1))x(m_1+1) - 2P(m_1)x(m_1) - \sum_{d=1}^{r} Q_d(m_1)x(\beta_d(m_1)) + h(m_1), m_1 \in [0,W-1] \\ x(0) = \varepsilon \in R \end{cases}$$

(6)where $h \in N([1,W],R)$ is bounded by Arzela–Ascoli's theorem.

Theorem 2

Suppose that $P, Q_d \in N([1,W],R_+), \beta_d \in N([0,W],[0,W]), \beta_d(m_1) \leq m_1, d = 1,2,...r.$ Assume $h \in N([0,W],R)$ and be bounded. Also assume (5)its also true. Then (6) has a unique solution.

Proof:

Suppose x is a solution of (6), then (6) can be written as

$$\Delta^2[x(m_1-1)\prod_{d=1}^{k_1-1}(1+P(d))] = [-\sum_{d=1}^{r} Q_d(d)x(\beta_d(m_1)) + h(m_1)]\prod_{d=1}^{m_1-1}(1+P(d)) \text{ for } m_1 \in [1,W].$$

Adding from 1 to $s-1$ we get,

$$\Delta(x(s)) = \left[\varepsilon + \sum_{d=1}^{s} \left(\left(-\sum_{u=1}^{r} \sum_{v=1}^{r} Q_v(d) x(\beta_m(d)) + h(d) \right) \right) \right] \prod_{d=1}^{s} [1+P(j)] (\prod_{d=1}^{s} [[1+P(j)])^{-1}$$

Adding from 0 to $r-1$ we get,

$$x(r) = \varepsilon + \sum_{0}^{r-1} \left[\varepsilon + \sum_{d=1}^{s} \left(\left(-\sum_{u=1}^{r} \sum_{v=1}^{r} Q_v(d) x(\beta_d(d)) + h(d) \right) \right) \right] \prod_{d=1}^{s} [1+P(j)] (\prod_{d=1}^{s} [[1+P(j)])^{-1}$$

$= [(A_h x)](s)$ for $s \in [0, W]$.

Moreover, if x is any solution of $x = A_h x$, then x is solution of (6).

Suppose (6) has two solutions a, b and $a \neq b$.

Let $\phi = a - b$ and $\phi(0) = 0$, and

$$\Delta^2 \phi(m_1 - 1) = \Delta^2 x(m_1 - 1) = -(P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1)) x(m_1+1) - 2P(m_1) x(m_1)$$
$$- \sum_{d=1}^{r} Q_d(m_1) x(\beta_d(m_1)) + h(m_1), m_1 \in [0, W-1]$$

By remark 1 $\phi(m_1) \leq 0$ for $m_1 \in [0, W]$. Therefore $a \leq b$ .Also if we consider $\phi = a - b,$ from remark 1we get, $b \leq a.$ Hence (6) has unique solution.

Theorem: 3

Let A_1 holds and x_0 and $y_0, x_0 \leq y_0$ be minimaland maximal values of solutions of (1). Assume A_3 :there exist functions $P, Q_d \in N([1, W], R_+), d = 1,2,..r$ such that (5) holds. Also

$$f_1(m_1, a, b_1, .., b_r) - f_1(m_1, \bar{a}, \bar{b}_1, ..., \bar{a}_r) \leq (P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))[\bar{a} - a]$$
$$-2P(m_1)[\bar{a} - a] - \sum_{d=1}^{r} Q_d(m)[\bar{b}_d - b_d]$$
$$x_0 \leq a \leq \bar{a} \leq y_0, \; x_0(\phi_d(t)) \leq b_d \leq \bar{b}_d \leq y_0(\phi_d(t)), \; d = 1,2,...r.$$

A_4 : f_2 isnon-increasing with respect to the second variable, and there exists a constant $b > 0$ such that,

$$f_2(\bar{a}, b) - f_2(a, b) \leq b(\bar{a} - a) \text{ for } x_0(0) \leq a \leq \bar{a} \leq y_0(0), x_0(W) \leq b \leq y_0(W)$$

ISSN: 2005-4238 IJAST

Then in $[x_0, y_0]_*$, (1) has minimal and maximal values of solutions

$$[x_0, y_0]_* = w \in \rho : x_0 \le w \le y_{0.}$$

$x(n) \le 0, n \in Z[0,T]$

Proof

Suppose we define f_2 by

$$(f_2(a,b))(m_1) = (P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))[\bar{a} - a] - 2P(m_1)[\bar{a} - a][a(m_1+1) - b(m_1+1)]$$
$$-2P(m_1)[a(m_1) - b(m_1)] + \sum_{d=1}^{r} Q_d(m_1)[a(\beta_d(m_1) - b(\beta_d(m_1))]$$

For $m_1 \in [1, W]$ and $k = 0,1,....,$ Consider

$$\begin{cases} \Delta^2 x_{k+1}(m_1 - 1) = F_1 x_k(m_1) - (f_2(x_{k+1} - x_k)(m_1) \\ x_{k+1}(0) = x_k(0) + \dfrac{1}{b} f_2(x_k(0), x_k(W) \end{cases}$$

$$\begin{cases} \Delta^2 y_{k+1}(m_1 - 1) = F_1 y_k(m_1) - (f_2(y_{k+1} - y_k)(m_1) \\ y_{k+1}(0) = y_k(0) + \dfrac{1}{b} f_2(y_k(0), y_k(W) \end{cases}$$

From theorem 2 x_1, y_1 are well defined. To prove that
$x_0 \le x_1 \le y_1 \le y_{0.}$ (7)

Let $\phi = x_0 - x_1$

$$\phi(0) = x_0(0) - x_0(0) + \frac{1}{b} f_2(x_0(0), x_0(W)) \le 0$$

$$\Delta^2 \phi(m_1 - 1) = F_1 x_0(m_1) - F_1 x_0(m_1) + f_2(x_1, x_0)(m_1)$$
$$= (P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))[a(m_1+1) - b(m_1+1)] - 2P(m_1)a(m_1) - b(m_1)]$$
$$-\sum_{j=t}^{k} \sum_{d=t}^{r} Q_d(m_1)\phi(\beta_d(m_1)) \qquad (8)$$

From (8) and Remark 1we get $\phi \le 0,$ therefore $x_0 \le x_1$.

To prove that $y_1 \le y_0$ and $x_1 \le y_1$.

Let $\phi = x_1 - y_1$. Then by using A_3 and A_4

$$\phi(0) = x_0(0) - y_0(0) + \frac{1}{b} f_2[y_0(0), y_0(W) - f_2(x_0(0), x_0(W))] \le 0$$

$$\Delta^2\phi(m_1 - 1) = F_1 x_0(m_1) - F_1 y_0(m_1) + f_2(x_1, x_0)(m_1) + f_2(y_1, y_0)(m_1)$$

$$\le (P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))\phi(m_1+1) - 2P(m_1)\phi(m_1) - \sum_{j=m_1}^{k}\sum_{d=m_1}^{r} Q_d(m_1)\phi(\beta_d(m_1))$$

Therefore (7) holds.

Bymathematical induction,

$x_0 \le x_1 \le ... \le x_k \le y_k \le ... \le y_1 \le y_0$.

$\{x_k\}$ is increasing and bounded, $\{x_k\}$ converging to x uniformly and $\{y_k\}$ is converging to y uniformly on $[0,W]$ and $x_o \le x \le y \le y_0$. f_1 and f_2 are continuous. Therefore x and y are solutions of equation(1).

To prove that x, y are minimal and maximal solutions of (1) in the sector $[x_0, y_0]_*$. Assume $a \in [x_0, y_0]$ any solution of (1) in that sector. Suppose that $x_i \le a \le y_i$ for some positive i. Let $\phi = x_i - a; \theta = a - y_i$. Then $\phi(0) \le 0, \theta(0) \le 0.$ and

$$\Delta^2\phi(m_1 - 1) = F_1 x_{i-1}(m_1) - f_2(x_i, x_{i-1})(m_1) - (F_1 a)(m_1)$$

$$\le (-P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))\phi(m_1+1) - 2P(m_1)\phi(m_1) - \sum_{j=m_1}^{k}\sum_{d=m_1}^{r} Q_d(m_1)\phi(\beta_d(m_1))$$

$$\Delta^2\theta(m_1 - 1) = (F_1 a)(m_1) - F_1 y_{d-1}(m_1) + f_2(y_d, y_{d-1})(m_1)$$

$$\le (-P(m_1)(P(m_1+1) + P(m_1) + P(m_1+1))\phi(m_1+1) - 2P(m_1)\phi(m_1) - \sum_{j=m_1}^{k}\sum_{d=m_1}^{r} Q_d(m_1)\theta(\beta_d(m_1)) \quad (9)$$

From (9) and remark 1we get $x_d \le a \le y_d$. By induction we get $x_k \le a \le y_k$. When n tends to infinity $x \le a \le y$. Hence the proved.

EXAMPLE:

Let $W \le 25$ and $0 < \varepsilon < 0.003$. We consider the following problem

ISSN: 2005-4238 IJAST

$$\begin{cases} \Delta^2 x(k_1-1) = -(\frac{4k_1}{1000}.\frac{4(k_1+1)}{1000}+\frac{4k_1}{1000}+(\frac{4(k_1+1)}{1000})x(k_1+1)-2(\frac{4k_1}{1000})x(k_1) \\ +\frac{2k_1}{1000}x(k_1-1)+\varepsilon = F_1(x)(k_1), k_1\in[1,W]. \\ x(0)=x(W) \end{cases} \quad (10)$$

So $f_2(a,b)=a-b.$ $\beta(0)=0$ and $\beta(k_1)=k_1-1$ if $k_1\in[1,W]$. Note that

$$P(k_1)=\frac{4k_1}{1000},\ Q_1(k_1)=\frac{2k_1}{1000},\ b=1$$

and

$$\delta_2=\sum_{k_1=1}^{W}\sum_{d=1}^{T}Q(r)\prod_{d=1}^{k_1-1}[1+P(m)]\le 1$$

$$\delta_2=\frac{1}{1000}\sum_{k_1=1}^{w}\sum_{d=1}^{T}2r_1\prod_{j=1}^{d-1}[1+\frac{4j}{1000}]\le\frac{50}{1000}\sum_{k_1=1}^{25}\sum_{d=1}^{15}(1+\frac{100}{1000})^{t-1}$$

$$=\frac{1}{20}[(1.1)^{25}-1]\approx 0.491735297\le 1$$

Therefore the assumption $A_1, A_{3,} A_4$ are satisfied.Let $x_0(k_1)=0, y_0(k_1)=1, k_1\in[1,W]$. Hence x_0, y_0 are minimal and maximal values of solution (10). By theorem 3, (10) has extremal solution in $[x_o, y_0]_*$.

References

1. Saber Elaydi,Anintroductiontodifferenceequations, Springer Science & Business Media, 2005.
2. Ravi P.Agarwal, Difference equations and inequalities theory, methods, and applications, Second edition, Revised and Expanded, Monongraphs,Pure and applied mathematics.
3. V. Lakshmikantham, D. Trigiante, Theory of difference equations. Numerical methods and applications, in: Mathematics in Science and Engineering, vol.181, AcademicPress, Inc., Boston, MA, 1988.
4. D. Jiang, J. Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, Nonlinear Anal. 50 (2002) 885–895.

5. TadeuszJankowski,First-orderfunctionaldifferenceequationswithnonlinearboundary valueproblems, ComputersandMathematicswithApplications59(2010)1937–1943.
6. A. P. Lavanya and Daphy Louis Lovenia J, "Positive Solutions of afuzzy nonlinear difference equations" Tamsui Oxford journal of information and mathematical sciences, Vol32(2), 2018.
7. Selvaraj B and Daphy Louis Lovenia J,Oscillation Behavior of fourth order neutral difference equations with variable coefficients, Far East Journal of Mathematical Sciences, 35(2), (2009), 225-231.
8. S. Sindhuja, J. Daphy Louis Lovenia, A.P. Lavanya, D. Darling Jemima, Qualitative Analysis of solutions of system of difference equations, Test engineering and management,ISSN: 0193 -4120 Page No. 15005-15007, January- February 2020.