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# New Coupled Fixed-Point Results in Partially Ordered Metric Space for Contractive Mappings

Qazi Aftab Kabir,
*Department of Mathematics, Govt. Degree College Surankote, Jammu &Kashmir, India*
Mayanglambam Siddhartha,
*Undergraduate Student, Department of Mathematics, Amity University Kolkata W.B, India*
Ramakant Bhardwaj,
*Associate Professor, Department of Mathematics, Amity University Kolkata W.B, India*
Pankaj Kumar Mishra
*Associate Professor, Department of Applied Physics AMITY University Gwalior (M.P.)*

***Abstract***

*In this paper, we prove some new fixed-point theorems for contractive mappings in partially ordered metric space which generalize the common unique fixed-point theorem to the case of multivalued mappings in partially ordered metric space with closed bounded set. Our results are the extensions of the results of some well-known recent result in the literature. Example is given showing that our result is proper extensions of the existing ones.*

*Keywords: contractive mappings, coupled fixed point (CFP), partial metric space (PMS), partially ordered metric space (POMS).*

## 1. Introduction

The idea of coupled fixed point was presented by Chang and Ma [8]. From that point forward, the idea has been important to numerous analysts in metrical fixed point hypothesis. Bhaskar and Lakshmikantham [3] built up coupled fixed point theorems in a metric space enriched with partial ordered request by utilizing the accompanying contractivity condition. Alaeidizaji et al. [1] and Bhardwaj [4] have recently demonstrated some new outcomes for withdrawals inCPMS. Wadker et al. [14] likewise demonstrated coupled fixed point hypothesis in somewhat requested PMS.Several papers have been published containing fixed point results for contractive mappings with different CBS in POMS(see[ 2, 3, 4 6, 7, 8 and 9]).

In this paper, we shall prove newcoupled fixed point theorems in partially ordered metric space with closed bounded set by employing somenotions of wadker et al. [15] as well as a rational type contractivecondition.

## 2. Definitions and Preliminaries

Throughout this paper we represent coupled fixed point **(CFP)**, Closed Bounded set **(CBS)**, complete partial metric space **(CPMS)**, partial metric space **(PMS)**, partially ordered metric space **(POMS)**.

Now, let us recall some basic concepts and facts about **(POMS)**.

**Definition 2.1**.[15]A partially ordered set is a set p and a binary relation ≤ denoted by $(X, \leq)$ such that for all $x,y,z \in p$

i. $x \leq x$, (reflexivity)
ii. $x \leq y,\ y \leq z \Longrightarrow x \leq z$ (transitivity)
iii. $x \leq y,\ y \leq x \Longrightarrow x = y$ (anti-symmetry)

**Definition 2.2.**[2]let (X, p) be a PMS

i. A sequence $\{x_n\}$ in (X, p) is said to convergence to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to} p(x, x_n)$.
ii. A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if $\lim_{n,m\to} p(x_n, x_m)$. exists and is finite.



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**Definition 2.3.** [14]A metric space(X, d) is said to be complete if every Cauchy sequence in X is convergent.

**Definition 2.4.**[3]Let(X, d) be a metric space. An element (X, y) ∈ X×X is said to be a CFPmapping F:X×X→X if F(x,y) = x and F(y,x) = y

**Definition 2.5.** [2]. A contractive mapping T:X→CB(X) is called a contraction mapping if there exists k ∈ (0,1) such that H(Tx,Ty) ≤ kd(x,y) ∀ x,y ∈ X and x ∈ Xis said to be a fixed point of T if x ∈ T(X)

## 3. Main Results

**Theorem 3.1.** Let (X, ≤) be a POMS and d be a metric on X such that (X, d) is complete metric space and let mappings T, S: X→C(X)

Satisfy the following conditions;

i) For each x ∈ X, T(x), S(x) ∈ CB(X),

ii) $H(T(x),S(y)) \leq \alpha_1[d(x,T(x))+d(y,S(y))] + \alpha_2[d(x, S(y))+d(y,T(x))]$

where $\alpha_1$, $\alpha_2$ are non-negative real numbers and $\alpha_1 + \alpha_2 < \frac{1}{2}$. Then there exists p ∈ X such that p ∈ T(x) ∩ S(x).

**Proof.** Let $x_o \in X$,$T(x_o)$ is a non-empty CBS of X. We can choose that $x_1 \in T(x_o)$,for this $x_1$by the same reason mentioned above $S(x_1)$ is non-empty CBS of X.

Since $x_1 \in T(x_o)$ and $S(x_1)$ are CBS of X, there exist $x_2 \in S(x_1)$ such that

$d(x_1,x_2) \leq H(T(x_o),S(x_1)) + \lambda$, where $\lambda = \max\left\{\frac{\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)},\frac{\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)}\right\}$

$$d(x_1,x_2) \leq H(T(x_o),S(x_1)) + \lambda$$

$$\leq \alpha_1[d(x_o,T(x_o))+d(x_1,S(x_1))] + \alpha_2[d(x_o, S(x_1))+d(x_1,T(x_o))] + \lambda$$

$$\leq \alpha_1[d(x_o,x_1)+d(x_1,x_2)] + \alpha_2[d(x_o, x_2)+d(x_1, x_1)] + \lambda$$

$$\leq \alpha_1[d(x_o,x_1)+d(x_1,x_2)] + \alpha_2[d(x_o, x_1)+d(x_1,x_2)] + \lambda$$

$$d(x_1,x_2) \leq \frac{\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)}d(x_o,x_1) + \lambda$$

$$d(x_1,x_2) \leq \lambda d(x_o,x_1) + \lambda$$

Thus for this $x_2$, $T(x_2)$ is a non-empty CBS of X.

Since $x_2 \in S(x_1)$ and $S(x_1)$ and $T(x_2)$ are CBS of X, there exist $x_3 \in T(x_2)$

Such that

$$d(x_2,x_3) \leq H(T(x_2),S(x_1)) + \lambda^2$$

$$\leq \alpha_1[d(x_2,T(x_2))+d(x_1,S(x_1))] + \alpha_2[d(x_2, S(x_1))+d(x_1,T(x_2))] + \lambda^2$$

$$\leq \alpha_1[d(x_2,x_3)+d(x_1,x_2)] + \alpha_2[d(x_2, x_2)+d(x_1, x_3)] + \lambda^2$$

$$\leq \alpha_1[d(x_2,x_3)+d(x_1,x_2)] + \alpha_2[d(x_1, x_2)+d(x_2, x_3)] + \lambda^2$$

$$d(x_2,x_3) \leq \frac{\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)}d(x_1,x_2) + \lambda^2$$

$$\leq \lambda d(x_1,x_2) + \lambda^2$$



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$\leq \lambda\{\lambda d(x_o,x_1)+\lambda\} + \lambda^2$

$$d(x_2,x_3) \leq \lambda^2 d(x_o,x_1) + 2\lambda^2$$

Similarly this process continue and we get a sequence $\{x_n\}$ such that $x_{n+1} \in S(x_n)$ or $x_{n+1} \in T(x_n)$ and $d(x_{n+1},x_n) \leq \lambda^n d(x_o,x_1) + n\lambda^n$.

Suppose $0 \ll u$ be given, choose that, a natural number $N_1$ such that $\lambda^n d(x_o,x_1) + n\lambda^n \ll u \ \forall\ n \geq N_1$

$$\Rightarrow d(x_{n+1},x_n) \ll u.$$

$\therefore \{x_n\}$ is a Cauchy sequence in (X,d) is a CPMS, $\exists\ p \in X$ such that $x_n \rightarrow p$. So choose a natural number $N_2$ such that

$d(x_n,p) \ll \dfrac{u(1-(\alpha_1+\alpha_2))}{2v(1+(\alpha_1+\alpha_2))}$ and

$d(x_{n-1},p) \ll \dfrac{u(1-(\alpha_1+\alpha_2))}{2v(\alpha_1+\alpha_2)} \forall\ n \geq N_2$.

$$d(T(p),p) \leq d(p,x_n) + d(x_n,T(p))$$

$\leq d(p,x_n) + H(S(x_{n-1}),T(p))$

$\leq d(p,x_n) + \alpha_1[d(x_{n-1},S(x_{n-1}))+d(p,T(p))] + \alpha_2[d(x_{n-1}, T(p))+d(p,S(x_{n-1}))]$

$\leq d(p,x_n) + \alpha_1[d(x_{n-1},x_n)+d(p,T(p))] + \alpha_2[d(x_{n-1}, T(p))+d(p,x_n)]$

$\leq d(p,x_n) + \alpha_1[d(x_{n-1},p)+d(p,x_n)+d(p,T(p))]$

$+\alpha_2[d((x_{n-1},p)+(p, T(p))+d(p,x_n)]$

$d(T(p),p) \leq \dfrac{\alpha_1+\alpha_2}{(1-(\alpha_1+\alpha_2)}d(x_{n-1},p) + \dfrac{(1+(\alpha_1+\alpha_2)}{(1-(\alpha_1+\alpha_2)} d(x_n,p) \forall\ n \geq N_2$.

$d(T(p),p) \ll \dfrac{u}{v}$ for all $v \geq 1$, we get $\dfrac{u}{v} - d(T(p),p) \in P$ and as $n \rightarrow$ , we get $\dfrac{u}{v} \rightarrow 0$ and P is closed $-d(T(p),p) \in P$ but $d(T(p),p) \in P$. Therefore $d(T(p),p) = 0$ and so $p \in T(p)$.

Similarly it can be established that $p \in S(p)$. Hence $p \in T(p) \cap S(p)$.

**Corollary 3.2.** Let (X, d) be a CPMS and let mappings $T_1, T_2: X \rightarrow C(X)$

Satisfy the following conditions;

i) For each $x \in X$, $T_1(x),T_2(x) \in CB(X)$,

ii) $H(T_1(x),T_2(y)) \leq a[d(x,T_1(x))+d(y,T_2(y))]$

Where a are non-negative real numbers and $a < \frac{1}{2}$. Then there exists $p \in X$ such that $p \in T_1(x) \cap T_2(x)$.

**Proof.** By substituting $T = T_1$, $S = T_2$, $\alpha_1 = a$ and $\alpha_2 = 0$ in the above theorem (3.1), it can be easily proved.

**Theorem 3.3.** Let (X, d) be a CPMS and let mappings $T, S: X \rightarrow C(X)$

satisfy the following conditions;

i) For each $x \in X$, $T(x), S(x) \in CB(X)$,

ii) $H(T(x),S(y)) \leq \alpha_1[d(x,y))+d(y,S(y))] + \alpha_2[d(y, S(y))+d(x,T(x))] + \alpha_3[d(y, T(x))+d(x,S(y))]$



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where $\alpha_1$, $\alpha_2$,$\alpha_3$ are non-negative real numbers and $\alpha_1 + \alpha_2 + \alpha_3 < \frac{1}{2}$. Then there exists $p \in X$ such that $p \in T(x) \cap S(x)$.

**Proof.** Let $x_o \in X$,$T(x_o)$ is a non-empty CBS of X. We can choose that $x_1 \in T(x_o)$,for this $x_1$by the same reason mentioned above $S(x_1)$ is non-empty closed bounded subset of X.

Since $x_1 \in T(x_o)$ and $S(x_1)$ are CBS of X, there exist $x_2 \in S(x_1)$ such that

$d(x_1,x_2) \le H(T(x_o),S(x_1)) + \lambda$, where $\lambda = \max\left\{\frac{\alpha_1+\alpha_2+\alpha_3}{1-(\alpha_1+\alpha_2+\alpha_3)},\frac{\alpha_1+\alpha_2+\alpha_3}{1-(\alpha_1+\alpha_2+\alpha_3)}\right\}$

$$d(x_1,x_2) \le H(T(x_o),S(x_1)) + \lambda$$

$$\le \alpha_1[d(x_o,x_1)+d(x_1,S(x_1))] + \alpha_2[d(x_1, S(x_1))+d(x_o,T(x_o))]$$

$$+\alpha_3[d(x_1, T(x_o))+d(x_o,S(x_1))] + \lambda$$

$$\le \alpha_1[d(x_o,x_1)+d(x_1,x_2)] + \alpha_2[d(x_1, x_2)+d(x_0, x_1)]$$

$$+\alpha_3[d(x_1, x_1)+d(x_o,x_2)] + \lambda$$

$$\le \alpha_1[d(x_o,x_1)+d(x_1,x_2)] + \alpha_2[d(x_1, x_2)+d(x_0, x_1)]$$

$$+\alpha_3[d(x_1, x_1)+d(x_o,x_1)+d(x_1,x_2)] + \lambda$$

$$d(x_1,x_2) \le \frac{\alpha_1+\alpha_2+\alpha_3}{1-(\alpha_1+\alpha_2+\alpha_3)}d(x_o,x_1) + \lambda$$

$$d(x_1,x_2) \le \lambda d(x_o,x_1) + \lambda$$

Thus for this $x_2$, $T(x_2)$ is a non-empty CBS of X.

Since $x_2 \in S(x_1)$ and $S(x_1)$ and $T(x_2)$ are CBS of X, there exist $x_3 \in T(x_2)$

such that

$$d(x_2,x_3) \le H(T(x_2),S(x_1)) + \lambda^2$$

$$\le \alpha_1[d(x_2,x_1)+d(x_1,S(x_1))] + \alpha_2[d(x_1, S(x_1))+d(x_2,T(x_2))]$$

$$+\alpha_3[d(x_1, T(x_2))+d(x_2,S(x_1))] + \lambda^2$$

$$\le \alpha_1[d(x_2,x_1)+d(x_1,x_2)] + \alpha_2[d(x_1, x_2)+d(x_2, x_3)]$$

$$+\alpha_3[d(x_1, x_3)+d(x_2,x_2)] + \lambda^2$$

$$\le \alpha_1[d(x_2,x_1)+d(x_1,x_2)] + \alpha_2[d(x_1, x_2)+d(x_2, x_3)]$$

$$+\alpha_3[d(x_1, x_2)+d(x_2, x_3)+d(x_2,x_2)] + \lambda^2$$

$$d(x_2,x_3) \le \frac{\alpha_1+\alpha_2+\alpha_3}{1-(\alpha_1+\alpha_2+\alpha_3)}d(x_1,x_2) + \lambda^2$$

$$\le \lambda d(x_1,x_2) + \lambda^2$$

$$\le \lambda\{\lambda d(x_o,x_1)+\lambda\} + \lambda^2$$

$$d(x_2,x_3) \le \lambda^2 d(x_o,x_1) + 2\lambda^2$$

Similarly this process continue and we get a sequence $\{x_n\}$ such that $x_{n+1} \in S(x_n)$ or $x_{n+1} \in T(x_n)$ and $d(x_{n+1},x_n) \le \lambda^n d(x_o,x_1) + n\lambda^n$.

Suppose $0 \ll u$ be given, choose that, a natural number $N_1$ such that $\lambda^n d(x_o,x_1) + n\lambda^n \ll u\ \forall\ n \ge N_1$



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$\therefore \{x_n\}$ is a Cauchy sequence in (X,d) is a CPMS, $\exists$ p $\in$ X such that $x_n \to p$. So choose a natural number $N_2$ such that

$d(x_n,p) \ll \frac{u(1-(\alpha_1+\alpha_2+\alpha_3))}{2v(1+(\alpha_1+\alpha_2+\alpha_3))}$ and

$d(x_{n-1},p) \ll \frac{u(1-(\alpha_1+\alpha_2+\alpha_3))}{2v(\alpha_1+\alpha_2+\alpha_3)} \forall\, n \geq N_2.$

$$d(T(p),p) \leq d(p,x_n) + d(x_n,T(p))$$

$$\leq d(p,x_n) + H(S(x_{n-1}),T(p))$$

$$\leq d(p,x_n) + \alpha_1[d(x_{n-1},p)+d(p,T(p))] + \alpha_2[d(p,T(p))+d(x_{n-1},S(x_{n-1}))]$$
$$+\alpha_3[d(p,S(x_{n-1}))+d(x_{n-1},T(p))]$$

$$\leq d(p,x_n) + \alpha_1[d(x_{n-1},p)+d(p,T(p))] + \alpha_2[d(p,T(p))+d(x_{n-1},x_n)]$$
$$+\alpha_3[d(p,x_n)+d(x_{n-1},T(p))]$$

$$\leq d(p,x_n) + \alpha_1[d(x_{n-1},p)+d(p,T(p))] + \alpha_2[d(p,T(p))+d(x_{n-1},p)+d(p,x_n)]$$
$$+\alpha_3[d(p,x_n)+d((x_{n-1},p)+(p,T(p))]$$

$$d(T(p),p) \leq \frac{\alpha_1+\alpha_2+\alpha_3}{(1-(\alpha_1+\alpha_2+\alpha_3)}d(x_{n-1},p) + \frac{(1+(\alpha_2+\alpha_3)}{(1-(\alpha_1+\alpha_2+\alpha_3)} d(x_n,p) \forall\, n \geq N_2.$$

$d(T(p),p) \ll \frac{u}{v}$ for all $v \geq 1$, we get $\frac{u}{v} - d(T(p),p) \in P$ and as $n \to$ , we get $\frac{u}{v} \to 0$ and P is closed $-d(T(p),p) \in P$ but $d(T(p),p) \in P$. Therefore $d(T(p),p) = 0$ and so $p \in T(p)$.

Similarly it can be established that $p \in S(p)$. Hence $p \in T(p) \cap S(p)$.

**Corollary 3.4.** Let (X, d) be a CPMS and let mappings $T_1, T_2: X \to C(X)$

satisfy the following conditions;

i) For each $x \in X$, $T_1(x), T_2(x) \in CB(X)$,

ii) $H(T_1(x),T_2(y)) \leq a[d(x,y))+d(y,T_2(y))] + c[d(y,T_1(x))+d(x,T_2(y))]$

Where a and c are non-negative real numbers and $a + c < \frac{1}{2}$. Then there exists $p \in X$ such that $p \in T_1(x) \cap T_2(x)$.

**Proof.** By substituting $T = T_1$, $S = T_2$, $\alpha_1 = a$, $\alpha_2 = 0$ and $\alpha_3 = c$ in the above theorem (3.3), it can be easily proved.

### Conflict of Interests

The authors declare that there is no conflict of interests.

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