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Feket-Szegö Inequalities for a new generalized class of Analytic functions

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Abstract

Here for a new unified class of analytic functions on the open unit disk $|z_1| < 1$ *the authors introduce Feket-Szegö inequalities . The sharp upperbounds of* $|a_3 - \eta a_2^2|$ *for the function* $h(z_1)$ *defined in the open unit disk is also discussed in this article. Special cases and corollary are elaborated by using main result which sows that the concluded results are more accurate and computable from standard result.*

Keywords: *Analytic functions, Feket-Szegö Inequalities, Upperbounds, Unit disk, Univalent functions*

1. INTRODUCTION

Let h is defined in the class A as -

$$h(z_1) = z_1 + \sum_{n=2}^{\infty} a_n z_1^n \tag{1.1}$$

it is analytic and univalent in $U = z_1 \in C : |z_1| < 1$ and normalized by $h(0) = 0$ and $h'(0) = 1$. All functions in A which are univalent in U are contained in a generalized class S.(see [1 - 3]).

Definition 1.1 (see also[1,4]) Let $S^*(\psi)$ and $C(\psi)$ are the class of starlike functions and convex functions respectively, defined as-

$$S^*(\psi) = Re\left\{\frac{z_1 h'(z_1)}{h(z_1)}\right\} \prec \psi(z_1), z_1 \in U \tag{1.2}$$

$$C(\psi) = Re\left\{1 + \frac{z_1 h''(z_1)}{h(z_1)}\right\} \prec \psi(z_1), z_1 \in U \tag{1.3}$$

where $\prec$ is the well known subordination between analytic functions. For detail one can see [2,3]

In this paper we will define a class $S^*C(\psi, \beta_1, \gamma_1)$ which is a generalization of the classes $S^*(\psi, \beta_1)$ and $C(\psi, \beta_1)$ introduced by Mustafa[3].

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Definition 1.2 Let a univalent starlike function with respect to '1' denoted by $\psi(z_1) = 1 + T_1 z_1 + T_2 z_1^2 + ...$, where $T_1 > 0$. The function $h \in S^*C(\psi, \beta_1, \gamma_1)$ if

$$\left\{\frac{z_1 h'(z_1) + \gamma_1 z_1^2 h''(z_1)}{\gamma_1 z_1[h'(z_1) + \beta_1 z_1 h''(z_1)] + (1-\gamma_1)[\beta_1 h'(z_1) + (1-\beta_1)h(z_1)]}\right\} = \psi(z_1) \quad (\alpha_1, \beta_1 \in [0,1), \gamma_1 \in [0,1) \tag{1.4}$$

When

$$\psi(z_1) = \frac{(1 + L z_1)}{(1 + M z_1)}, (-1 \leq M < L \leq 1),$$

Here in this article , we will attain the Fekete-Szegö inequality for $h(z_1)$ in the above discussed class $S^*C(\psi, \beta_1, \gamma_1)$

Lemma 1.1 [8]

If $p_1(z_1) = 1 + d_1 z_1 + d_2 z_1^2 + ...$ and $\eta \in C$

$|d_2 - \eta d_1^2| \leq 2max\{1, |2\eta - 1|\}$ and for $p_1(z_1) = \frac{1+z_1}{1-z_1}$ and $p_1(z_1) = \frac{1+z_1^2}{1-z_1^2}$

Lemma 1.2 If $p_1(z_1) = 1 + d_1 z_1 + d_2 z_1^2 + ...$ and $\xi \in R$, $\eta \in C$

$$|d_2 - \xi d_1^2| \leq \begin{cases} -4\xi + 2, & \xi \leq 0 \\ 2, & 0 \leq \xi \leq 1 \\ 4\xi - 2, & \xi \geq 1 \end{cases}$$

When $\xi < 0$ or $\xi > 1$, the equality holds iff $p_1(z_1)$ is $\dfrac{1+z_1}{1-z_1}$ or one of its revolutions. If $0 < \xi < 1$, then the equality holds iff $p_1(z_1)$ is $\dfrac{1+z_1^2}{1-z_1^2}$ or one of its revolutions. If $\xi = 0$, the equality holds iff

$$p_1(z_1) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z_1}{1-z_1} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z_1}{1+z_1} \quad (0 \leq \lambda \leq 1)$$

or one of its revolutions. If $\xi = 1$, the equality holds iff $p_1(z_1)$ is the reciprocal of one of its functions such that the equality holds in the case of $\xi = 0$. Also the above upper bound is sharp, and can be improved in the following manner when $0 < \xi < 1$

$$|d_2 - \xi d_1^2| + \xi |d_1|^2 \leq 2 \qquad (0 < \xi \leq 1/2)$$

$$|d_2 - \xi d_1^2| + (1-\xi)|d_1|^2 \leq 2 \qquad (1/2 < \xi < 1)$$

2. MAIN RESULTS

Theorem 2.1 If $h(z_1) \in S^*C(\psi, s, t)$ and $\eta \in C$, then

$$|a_3 - \eta a_2^2| \leq \lambda\{T_1, T^*\}$$

where

$$\lambda = \frac{1}{2(1 + 2\gamma_1 - \beta_1 - \beta_1\gamma_1)} \tag{2.1}$$

and

$$T^* = \left| T_2 + T_1^2 \frac{(1+\beta_1+\gamma_1+\beta_1\gamma_1)}{(1-\beta_1+\gamma_1-\beta_1\gamma_1)} + 2\eta T_1^2 \frac{(1+\beta_1+\gamma_1+\beta_1\gamma_1)}{(1-\beta_1+\gamma_1-\beta_1\gamma_1)} \right| \tag{2.2}$$

provided $1-\beta_1+\gamma_1-\beta_1\gamma_1 \neq 0$ and $1-\beta_1+2\gamma_1-\beta_1\gamma_1 \neq 0$. This result is sharp.

Proof Let $h \in S^*C(\psi, \beta_1, \gamma_1)$ and by using the *Schwarz* function $w(z_1) \in A$ we get

$$\left\{ \frac{z_1 h'(z_1) + \gamma_1 z_1^2 h''(z_1)}{\gamma_1 z_1[h'(z_1) + \beta_1 z_1 h''(z_1)] + (1-\gamma_1)[\beta_1 h'(z_1) + (1-\beta_1)h(z_1)]} \right\} = \psi(w(z_1)) \quad (z_1 \in U) \tag{2.3}$$

If $p_2(z_1)$ is obtained from lemma 1.1 with the same condition of having $p_1(0) = 1$, then

$$p_2(z_1) = \frac{1+w(z_1)}{1-w(z_1)} = 1 + d_1 z_1 + d_2 z_1^2 + ... \quad (z_1 \in U) \tag{2.4}$$

by using (2.2) in the above equations we get

$$w(z_1) = \frac{d_1}{2} z_1 + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) z_1^2 + ... \tag{2.5}$$

also

$$p_1(z_1) = \left\{ \frac{z_1 h'(z_1) + \gamma_1 z_1^2 h''(z_1)}{\gamma_1 z_1[h'(z_1) + \beta_1 z_1 h''(z_1)] + (1-\gamma_1)[\beta_1 h'(z_1) + (1-\beta_1)h(z_1)]} \right\} = 1 + b_1 z_1 + b_2 z_1^2 + ... \tag{2.6}$$

$$(z_1 \in U)$$

which leads to

$$b_1 = (1+\gamma_1-\beta_1-\beta_1\gamma_1)a_2 \quad and \quad b_2 = (1+\beta_1+\gamma_1+\beta_1\gamma_1)(1-\beta_1+\gamma_1-\beta_1\gamma_1)a_2^2 + 2(1+2\gamma_1-\beta_1-\beta_1\gamma_1)a_3 \tag{2.7}$$

by the definition of subordination and (2.4), we get:

$$p_1(z_1) = \psi(w(z_1)) = 1 + \frac{T_1 d_1}{2} + \left\{ \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) T_1 + \frac{1}{4} d_1^2 T_2 \right\} z_1^2 + ... \quad (z_1 \in U) \tag{2.8}$$

Now from (2.6),(2.7) and (2.8), we have

$$(1+\gamma_1-\beta_1-\beta_1\gamma_1)a_2 = \frac{T_1 d_1}{2}$$

$$\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) T_1 + \frac{1}{4} d_1^2 T_2 = (1+\gamma_1+\beta_1+\beta_1\gamma_1)(1-\beta_1+\gamma_1-\beta_1\gamma_1)a_2^2 + 2(1+2\gamma_1-\beta_1-\beta_1\gamma_1)a_3$$

Therefore we have

$$a_3 - \eta a_2^2 = \frac{T_1}{4(1+2\gamma_1-\beta_1-\beta_1\gamma_1)} \left\{ d_2 - \xi d_1^2 \right\}, \quad (1+2\gamma_1-\beta_1-\beta_1\gamma_1 \neq 0), (1+\gamma_1-\beta_1-\beta_1\gamma_1 \neq 0) \tag{2.9}$$

where

$$\xi = \frac{1}{2}\left\{ 1 - \frac{T_2}{T_1} - \left(\frac{1+\gamma_1+\beta_1+\beta_1\gamma_1}{1-\beta_1+\gamma_1-\beta_1\gamma_1} \right) T_1 + \left(\frac{1+2\gamma_1-\beta_1-\beta_1\gamma_1}{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2} \right) 2\eta T_1 \right\}$$

$$(1+2\gamma_1-\beta_1-\beta_1\gamma_1\neq 0),(1+\gamma_1-\beta_1-\beta_1\gamma_1\neq 0)$$

the result is sharp for

$$\left\{\frac{z_1h'(z_1)+\gamma_1 z_1^2h''(z_1)}{\gamma_1z_1[h'(z_1)+\beta_1z_1h''(z_1)]+(1-\gamma_1)[\beta_1h'(z_1)+(1-\beta_1)h(z_1)]}\right\}=\psi(z_1) \tag{2.10}$$

and

$$\left\{\frac{z_1h'(z_1)+\gamma_1 z_1^2h''(z_1)}{\gamma_1z_1[h'(z_1)+\beta_1z_1h''(z_1)]+(1-\gamma_1)[\beta_1h'(z_1)+(1-\beta_1)h(z_1)]}\right\}=\psi(z_1^2) \tag{2.11}$$

Corollary 2.2 For $h(z_1)\in S^*C(\psi,\beta_1,\gamma_1)$, for real parameters η such that

$1+2\gamma_1-\beta_1-\beta_1\gamma_1\neq 0, 1+\gamma_1-\beta_1-\beta_1\gamma_1\neq 0$,then

$$|a_3-\eta a_2^2|\leq\begin{cases} T^* & \eta\leq\rho_1,\\ T_1 & \rho_1\leq\eta\leq\rho_2,\\ -T^* & \eta\geq\rho_2,\end{cases}$$

where

$$\rho_1=\frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{2T_1(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\left[-1+\frac{T_2}{T_1}+T_1\left(\frac{1+\beta_1+\gamma_1+\beta_1\gamma_1}{1+\gamma_1-\beta_1-\beta_1\gamma_1}\right)\right]$$
$$\rho_2=\frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{2T_1(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\left[1+\frac{T_2}{T_1}+T_1\left(\frac{1+\beta_1+\gamma_1+\beta_1\gamma_1}{1+\gamma_1-\beta_1-\beta_1\gamma_1}\right)\right]$$

By taking η to be real number and by taking lemma 1.2 we will get the desired result.

Remark 1: $K_\psi^n(n=2,3,.....)$ is defined to represent the sharpness of these bounds for real parameter η as follows:

$$\left\{\frac{z_1K'^n_\psi(z_1)+\gamma_1z_1^2K''^n_\psi(z_1)}{\gamma_1z_1\left[K'^n_\psi(z_1)+\beta_1z_1K'^n_\psi(z_1)\right]+(1-\gamma_1)\left[\beta_1K''^n_\psi(z_1)+(1-\beta_1)K^n_\psi(z_1)\right]}\right\}=\psi(z_1^{(n-1)})$$

$$K^n_\psi(0)=K'^n_\psi-1$$

and the functions F_λ and H_λ $(0\leq\lambda\leq 1)$ by

$$\left\{\frac{z_1F'_\lambda(z_1)+\gamma_1z_1^2F''_\lambda(z_1)}{\gamma_1z_1\left[F'_\lambda(z_1)+\beta_1z_1F''_\lambda(z_1)\right]+(1-\gamma_1)\left[\beta_1F''_\lambda(z_1)+(1-\beta_1)F_\lambda(z_1)\right]}\right\}=\psi\left(\frac{z_1(z_1+\lambda)}{1+\lambda z_1}\right)$$

$$F_\lambda(0)=F'_\lambda-1$$

$$\left\{\frac{z_1H'_\lambda(z)+\gamma_1z_1^2H''_\lambda(z)}{\gamma_1z_1\left[H'_\lambda(z)+\beta_1z_1H''_\lambda(z_1)\right]+(1-\gamma_1)\left[\beta_1H''_\lambda(z_1)+(1-\beta_1)H_\lambda(z_1)\right]}\right\}=\psi\left(\frac{z_1(z_1+\lambda)}{1+\lambda z_1}\right)$$

$$H_\lambda(0)=H'_\lambda-1$$

Obviously these functions $K^n_\psi, F_\lambda, H_\lambda\in S^*C(\psi,\beta_1,\gamma_1)$. If $\eta<\rho_1$ or $\eta>\rho_2$, then equality holds iff h is K^2_ψ or one of its revolutions. When $\rho_1<\eta<\rho_2$ then equality holds iff h is K^3_ψ

or one of its revolutions. If $\eta = \rho_1$ then equality holds iff h is F_λ or one of its revolutions . If $\eta = \rho_2$ then equality holds iff h is H_λ or one of its revolutions. If $\rho_1 \leq \eta \leq \rho_2$, Corollary 2.2 can be improved, taking in account Lemma 1.2.

Corollary 2.3 Let $h(z_1) \in S(\psi, s, t)$, for real parameters η such that
$1 + 2\gamma_1 - \beta_1 - \beta_1\gamma_1 \neq 0, 1 + \gamma_1 - \beta_1 - \beta_1\gamma_1 \neq 0$ and ρ_3 is given by

$$\rho_3 = \frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{T_1(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\left[\frac{T_2}{T_1} + T_1\frac{1+\beta_1+\gamma_1+\beta_1\gamma_1}{1+\gamma_1-\beta_1-\beta_1\gamma_1}\right]$$

If $\rho_1 < \eta \leq \rho_3$, then

$$|a_3 - \eta a_2^2| + \frac{1}{2(T_1^2)}\left[(T_1 - T_2)\frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}T_1^2\right.$$

$$\left.+T_1\left(\frac{(1+\beta_1+\gamma_1+\beta_1\gamma_1)(1+\gamma_1-\beta_1-\beta_1\gamma_1)}{(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\right) - 2\eta T_1^2\right] \leq \frac{T_1}{2(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}$$

If $\rho_3 < \eta \leq \rho_2$, then

$$|a_3 - \eta a_2^2| + \frac{1}{2T_1^2}\left[(T_1 + T_2)\frac{(1+\gamma_1-\beta_1-\beta_1\gamma_1)^2}{1+2\gamma_1-\beta_1-\beta_1\gamma_1}\right.$$

$$\left.+T_1^2\frac{(1+\beta_1+\gamma_1+\beta_1\gamma_1)(1+\gamma_1-\beta_1-\beta_1\gamma_1)}{(1+2\gamma_1-\beta_1-\beta_1\gamma_1)} - 2\eta T_1^2\right] \leq \frac{T_1}{2(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}$$

where ρ_1 and ρ_2 are same as given in Corollary 2.2.

Example 2.4 Let $(-1 \leq M < L \leq 1)$. If $h(z) \in S^*C[L, M, \beta_1, \gamma_1]$, for real parameter η, then

$$|a_3 - \eta a_2^2| \leq \ \lambda\begin{cases} C^* & \eta \leq \rho_1, \\ L - M & \rho_1 \leq \eta \leq \rho_2, \\ -C^* & \eta \geq \rho_2, \end{cases}$$

where

$$\rho_1 = \frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{2(L-M)(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\left[-1 + \frac{M(M-L}{L-M} + (L-M)\left(\frac{1+\beta_1+\gamma_1+\beta_1\gamma_1}{1+\gamma_1-\beta_1-\beta_1\gamma_1}\right)\right]$$

$$\sigma_2 = \frac{(1-\beta_1+\gamma_1-\beta_1\gamma_1)^2}{2(L-M)(1+2\gamma_1-\beta_1-\beta_1\gamma_1)}\left[-1 + \frac{M(M-L}{L-M} + (L-M)\left(\frac{1+\beta_1+\gamma_1+\beta_1\gamma_1}{1+\gamma_1-\beta_1-\beta_1\gamma_1}\right)\right]$$

λ is defined in (2.1) and

$$C^* = \left|M(M-L) + (L-M)^2\frac{(1+\beta_1+\gamma_1+\beta_1\gamma_1)}{(1-\beta_1+\gamma_1-\beta_1\gamma_1)} - 2\eta(L-M)^2\frac{(1-\beta_1+2\gamma_1-\beta_1\gamma_1)}{(1-\beta_1+\gamma_1-\beta_1\gamma_1)}\right|$$

Remark 2: For $\beta_1 = 0$ in aforementioned Corollaries 2.2 and Example 2.4 we get the results by Shanmugham et al. [6].

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