Limiting Behavior of Solutions of Second Order Linear Difference Equation

[1]V. Ananthan, [2]M. Ramesh

[1]Associate Professor, Department of Mathematics, School of Arts and Science, AVIT Campus, Paiyanoor, Chennai
[2]Assistant Professor, Vinayaka Mission's Research Foundation, Selam
[1]anandhanavit@gmail.com, [2]ramesh.maths@avit.ac.in

Abstract

We deliberate limiting performance of explanations of second order linear difference equation $y_{n+2} + p_1 y_{n+1} + p_2 y_n = 0 - - - - - - - - - - - - - (1)$

Keywords: *Difference equation, Limiting Behavior, Oscillation, Non-oscillation, Unbalanced scheme, Steady scheme.*

1. Introduction

Difference equations are the equivalent of differential equations. They stand up logically in corporeal problems when the explanation is isolated in nature. Hence the equations are copious in number. The difference equation of n^{th} order will be $f(x_t, \Delta x_t, \Delta^2 x_t, \ldots \ldots \ldots \ldots \ldots \Delta^n x_t) = 0$. Here Δ is the advancing difference operator defined by $\Delta x_t = x_{t+1} - x_t$. The second order difference equations presentation a essential character in many divisions of pure and applied mathematics such as sustained slight, singular roles, orthogonal polynomial, and combinatorics. In this paper we will utilize the special characteristics of second order equations to obtained a deeper understanding of the limiting behavior of solution of second order difference equation. To abridge our description, we control our conversation to the second-order difference equation of the system

$$y_{n+2} + p_1 y_{n+1} + p_2 y_n = 0 - - - - - - - - - - - - - (1)$$

Presume that $\lambda_1 \; and \; \lambda_2$ are the distinguishing roots of the equation.

2. Main Results

We have the ensuing three cases:

Case (1):

$\lambda_1 \; and \; \lambda_2$ are discrete actual roots. Then $y_1(n) = \lambda_1^n \; and y_2(n) = \lambda_2^n$ are two linearly self-governing explanations of (1). If $|\lambda_1| > |\lambda_2|$, formerly we demand $y_1(n)$ the leading answer, and $\lambda_1$the leading distinguishing origin. Otherwise $y_2(n)$ is the leading answer and λ_2 is the main distinctive origin.

We determination nowadays expression that the preventive performance of the overall answer $y(n) = a_1\lambda_1^n + a_2\lambda_2^n$ is strong-minded by the behavior of the leading solution. So, assume, without loss of generality, that $|\lambda_1| > |\lambda_2|$. Then

$$y(n) = y_1^n \left[a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n \right]$$

Meanwhile

$$\left|\frac{\lambda_2}{\lambda_1}\right| < 1$$

It surveys that

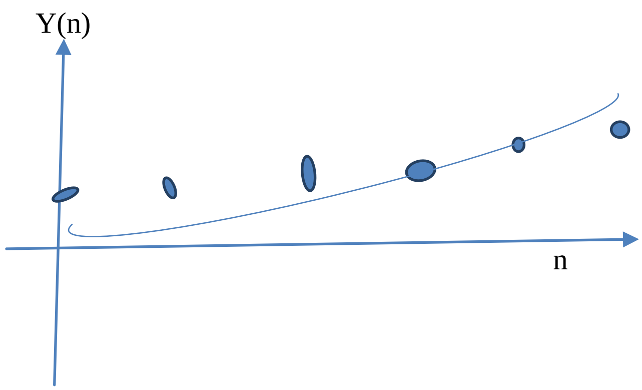
$$\left(\frac{\lambda_1}{\lambda_2}\right)^n \to 0 \; as \; n \to \infty$$

Accordingly,

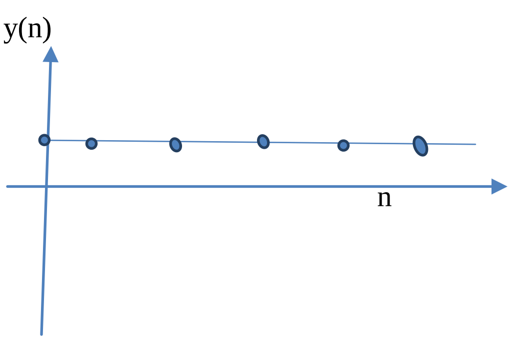
$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\lambda_1^n.$$

Here stand six unlike states that may arise here liable on the value of λ_1

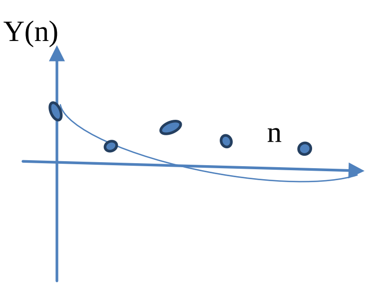
1. $\lambda_1 > 1$: *The sequence* $\{a_1\lambda_1^n\}$ *diverges to* ∞ (Unbalanced scheme).



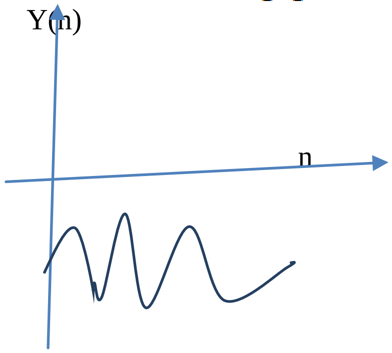
2. $\lambda_1 = 1$: *The sequence* $\{a_1\lambda_1^n\}$ *is a constant sequence.*



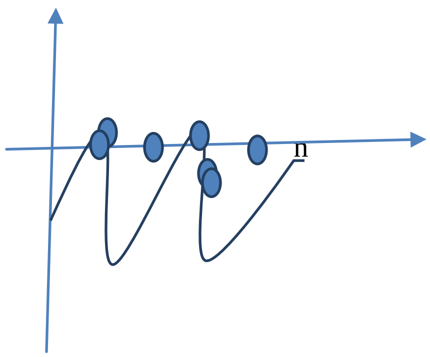
3. $0 < \lambda_1 < 1$: *The sequence* $\{a_1\lambda_1^n\}$ *is monotonically decreasing to zero* (Steady scheme).



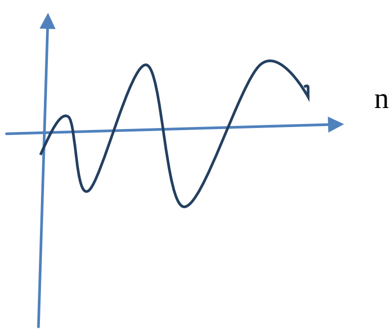
4. $-1 < \lambda_1 < 0$: *The sequence* $\{a_1\lambda_1^n\}$ *is oscillating around zero* (Steady scheme)



5. $\lambda_1 = -1$: *The sequence* $\{a_1\lambda_1^n\}$ *is oscillating between two values* a_1 *and* $-a_1$.



6. $\lambda_1 < -1$: *The sequence* $\{a_1\lambda_1^n\}$ *is oscillating but increasing in magnitude* (Unstable system).



Case (2):

If $\lambda_1 = \lambda_2 = \lambda$

The overall result of reckoning (1) is assumed by $y(n) = (a_1 + a_2 n)\lambda^n$.

Clearly, if $|\lambda| \geq 1$, *the solution* $y(n)$ *diverges either monotonically if* $\lambda \geq 1$ *or by oscillating if*

$\lambda < 1$, *then the solution converges to zero, since* $\lim_{n\to\infty} n\lambda^n = 0$.

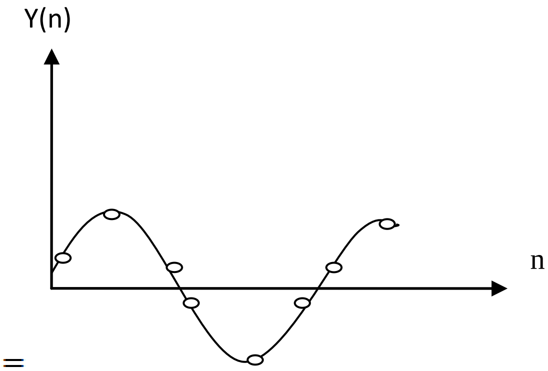
Case (3):

Complex roots: $\lambda_1 = \alpha + i\beta \text{ and } \lambda_2 = \alpha - i\beta, \text{where } \beta \neq 0.$ As we have the solution of (1) is given by $y(n) = ar^n \cos(n\theta - \omega),$ Where $r = \sqrt{\alpha^2 + \beta^2},\ \ \theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

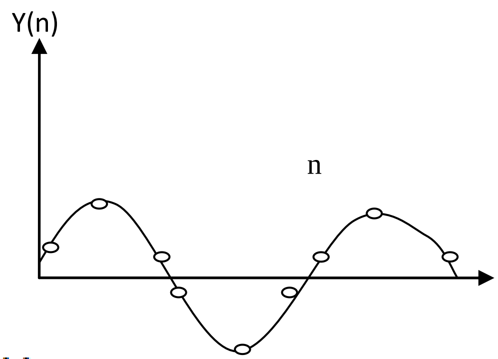
The solution $y(n)$ obviously hesitates, then the cos functions hesitates.

Still, $y(n)$ vacillates in three unlike conducts liable on the position of the conjugate individual origins, by way of can be realized in the ensuing numeral

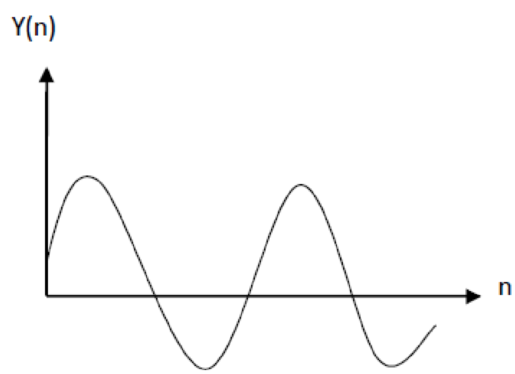
$1. r > 1: Here\ \lambda_1\ and\ \lambda_2 = \overline{\lambda_1}$ are outside the unit circle. Hence $y(n)$ is oscillating but increasing in magnitude (Unstable system).



$2. r = 1: Here\ \lambda_1\ and\ \lambda_2 = \overline{\lambda_1}\ lie\ on\ the\ unit\ circle. In\ this\ case\ y(n) is\ oscillating\ but\ constant\ in\ magnitude.$



$3. r < 1: Here\ \ \lambda_1\ and\ \lambda_2 = \overline{\lambda_1} lie\ inside\ the\ unit\ disk. The\ solution\ y(n) oscillates\ but\ converges\ to\ zero\ as\ n \to \infty$ (Even scheme).



In conclusion, we review the upstairs argument in the subsequent statement.

Theorem 1:

The next declarations grip in case of second order direct standardized difference equation

(i) Entirely result of (1) waver (about zero) if and only if the representative reckoning has no confident actual roots.
(ii) Entirely answers of (1) touches to nothing (that is the zero solution is asymptotically stable) gamble and lone if $max\{|\lambda_1|, |\lambda_2|\} < 1$.

Next, we study nonhomogeneous difference equations in which the contribution is persistent, that is, comparation of the method

$y_{n+2} + p_1 y_{n+1} + p_2 y_n = m$ -- (2)

Someplace m is a nonzero endless contribution or compelling period. Different (1), the nil classification $y(n) = 0 \; for \; all \; n\epsilon Z^+ \; is \; not \; a \; solution \; of \; (2).$

As a substitute, we have the balance idea or explanation $y(n) = y^*$.

Meanwhile (2) we revenue $y^* + p_1 y^* + p_2 y^* = m$

$y^* = \frac{m}{1+p_1+p_2}$ --- (3)

Thus $y_p(n) = y^* \; is \; a \; particular \; solution \; of \; (2)$. Accordingly, the overall resolution of (2) is agreed by

$y(n) = y^* + y_c(n)$ -- (4)

It is clear that $y(n) = y^* \; if \; and \; only \; if \; y_c(n) \to 0 \; as \; n \to \infty$. Furthermore, $y(n) oscillates \; about \; y^* \; if \; and \; only \; if \; y_c(n) \; oscillates \; about \; zero.$

These comments are brief in the succeeding theorem.

Theorem 2:

(i) Entirely results of the nonhomogeneous equation (2) hesitate about the uniformity solution y^* uncertainty and lone if nobody of the distinguishing origins of the Standardized calculation (1) is a constructive factual number.

(ii) Altogether keys of (2) joins to

$y^* \; as \; n \to \infty \; if \; and \; only \; if \; max\{|\lambda_1|, \; |\lambda_2|\} < 1,$
$where \; \lambda_1 and \; \lambda_2 \; are \; the \; characteristic \; roots \; of \; the \; homogeneous \; equation \; (1).$

Statements 1 and 2 bounce and adequate settings below which a following – instruction difference equation is asymptotically stable. In countless claims, however, one requirement to have clear standards for constancy founded on the morals of the coefficients p_1 and p_2 of (1) or (2). The subsequent outcome delivers us with such desired principles.

Theorem 3:

The Settings $1+p_1+p_2 > 0$, $1- p_1+p_2 >0$, $1-p_2 >0$ --------------------(5)are essential and satisfactory for the balance idea (key) of comparations (1) and (2) to be asymptotically stable (i.e., all solutions converge to y*).

Proof:

Accept that the symmetry opinion of (1) or (2) is asymptotically unchanging. In asset of Theorems 1 and 2 the origins λ_1, λ_2 of the distinguishing reckoning $\lambda^2 +p_1\lambda +p_2 = 0$ lie confidential the element disk,i.e.,$|\lambda_1|< 1$ and $|\lambda_2| <1$. By the quadratic formulary, we have

$$\lambda_1 = \frac{-p_1+\sqrt{p_1^2-4p_2}}{2} \text{ and } \lambda_2 = \frac{-p_1-\sqrt{p_1^2-4p_2}}{2} \qquad \text{--------------- (6)}$$

Formerly we have two belongings to contemplate.

Case 1:

λ_1, λ_2 are physical origins that is $p_1^2 - 4p_2 \geq 0.$ From formula (6) we have

$$-2 < -p_1 + \sqrt{p_1^2 - 4p_2} < 2$$

Or

$$-2 + p_1 < \sqrt{p_1^2 - 4p_2} < 2 + p_1 \text{ ---------------------------- (7)}$$

Also, one finds

$$-2 + p_1 < -\sqrt{p_1^2 - 4p_2} < 2 + p_1 \text{ --------------------------- (8)}$$

Shaping the second inequality in countenance (7) produces

$$1 + p_1 + p_2 > 0 \text{ -- (9)}$$

Likewise, if we four-sided the first inequality in appearance (8) we gain

$$1 - p_1 + p_2 > 0 \text{ --- (10)}$$

Nowadays after the additional dissimilarity of (7) besides the original variation of (8) we become
$2 + p_1 > 0 \text{ and } 2 - p_1 > 0 \text{ or } |p_1| < 2$

Since $p_1^2 - 4p_2 \geq 0, p_2 \leq \frac{p_1^2}{4} < 1.$ This finalizes the impermeable of (5) in this occasion.

Case 2:

$\lambda_1 \text{ and } \lambda_2$ are composite conjugates, that is $p_1^2 - 4p_2 < 0.$ In this case we have

ISSN: 2005-4238 IJAST

$\lambda_{1,2} = \frac{-p_1}{2} \mp \frac{i}{2}\sqrt{4p_2 - p_1^2}$

Moreover, since $p_1^2 < 4p_2, it\ follows\ that -2\sqrt{p_2 < p_1 < 2\sqrt{p_2}}.$

Now $|\lambda_1|^2 = \frac{p_1^2}{4} + \frac{4p_2}{4} - \frac{p_1^2}{4}$. Since $|\lambda_1| < 1$. It follows that $0 < p_2 < 1$.

Later to confirmation that the foremost binary disparities of (5) grip we essential to display that the purpose $f(x) = 1 + x - 2\sqrt{x} > 0\ for\ x\epsilon(0,1)$. Otherwise that $f(0) = 1, and\ f^1(x) = 1 - \frac{1}{\sqrt{x}}$. Thus x=1 is a local minimum as f(x) decreases for $x \in (0,1). Hence\ f(x) > 0 for\ all\ x \in (0,1)$. This finishes the resistant of the needed environments.

Example-1:

Discovery circumstances below which the result of the calculation

$$y(n+2) - \alpha(1+\beta)y(n+1) + \alpha\beta y(n) = 1, \qquad \alpha, \beta > 0.$$

(a) Meets to the symmetry point y^*, and
(b) Waver about y^*.

Solution:

Lease us initial discovery the symmetry opinion $y^*. Be\ letting\ y(n) = y^*\ in\ the\ equation.$

We obtain $\quad y^* = \frac{1}{1-\alpha}, \qquad \alpha \neq 1.$

(a). Applying condition (5) to our equation yields
$\alpha < 1, \qquad 1 + \alpha + 2\alpha\beta > 0, \qquad \alpha\beta < 1.$

Clearly, the second inequality

$1 + \alpha + 2\alpha\beta > 0\ is\ always\ satisfied, since\ \alpha, \beta\ are\ both\ positive\ numbers.$

(b). The explanations stand oscillatory around $y^*\ if\ either\ \lambda_1, \lambda_2$ remain undesirable actual facts or compound conjugates. Fashionable the primary case we consume

$$\alpha^2(1+\beta)^2 > 4\alpha\beta, \ \ or \qquad \alpha > \frac{4\beta}{(1+\beta)^2}$$

then $\quad \alpha(1+\beta) < 0,$

This is impossible. Thus if $\alpha > {}^{4\beta}\!/_{(1+\beta)^2}$, we have no oscillatory solutions.

Now $\lambda_1\ and\ \lambda_2$ multifaceted conjugates if $\alpha^2(1+\beta)^2 < 4\alpha\beta, \ \ or \qquad \alpha < \frac{4\beta}{(1+\beta)^2}$

Later all answers are oscillatory if $\qquad \alpha < \frac{4\beta}{(1+\beta)^2}$.

References

[1] I.Gyori and G.Ladas, Oscillation Theory of Delay differential Equation, Oxford science, oxford, (1991).
[2] R.D.Driver, G.Ladas and P.N.Viahos, Asymptotic behavior of a linear delay difference equation, proc.Amer, mathematics.Soc.115, 105-112, (1992).
[3] Bingtuan Li, Oscillations of Delay Differential Equations with Variable Coefficients, J. Math. Anal. Appl. 192(1995), 312-321.
[4] Chuan Jun Tian and B.G Zhang, New Oscillation Criteria for a Class of Linear Delay Differential Equations, Bull London Math. Soc. 31(1999), 196-206.
[5] S.G. Deo, V. Lakshmikantham and V. Ragavendra. Text Book of Ordinary Differential Equations, Tata McGraw Hill Pub. Co. New Delhi,1997.
[6] A. Elbert and I.P. Stavroulakis, Oscillation and Non Oscillation Criteria for Delay Differential Equations, Proc. Ameri. Math. Soc. 123(1995), 1503-1510.
[7] A. Elbert, Comparison Theorems for First Order Nonlinear Differential with Delay, Studia Sci. Math. Hungan.11(1976), 259267.
[8] L.H. Erbe, Qingkai Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations.
[9] L.H. Erbe and B.G. Zhang, Oscillation of First Order Linear Differential Equations with Deviating Arguments, Differential and Integral Equation I(1988), 305-314.
[10] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Oxford Sci. Publ. Clarendon Press, 1999.

ISSN: 2005-4238 IJAST