International Journal of Advanced Science and Technology
Vol. 29, No. 7s, (2020), pp. 1550-1556

CONTINUED FRACTIONS OFRATIOS OF POLYGONAL NUMBERS WITH CONSECUTIVE ORDERS AND RANKS

P.Balamurugan[1], A. Gnanam[2],R.Senthilkumar[3]Sivaranjani M [5]
[1] *Department of Science and Humanities, M.Kumarasamy College of Engineering(Autonomous), Karur – 639113.*
[2] *Department of Mathematics, Government Arts College, Trichy – 620022.*
[3] *Department of Mathematics Wing,DDE,Annamalai University*,Chidambaram - 608002.

Abstract

In number theory, study of number sequences is an exciting field. Among these the sequenceof figurate numbers provide a separate abundance in its suitability.Figurate number which have both order and rank of differentdimensions. Here, the study is only on two dimensional approach of figurate numbers.The ratios of polygonal numbers are studied as continued fractions. In this studysequence of continued fraction which produce ratios of figurate number of consecutive orders and ranks are classified.

Keywords: *continued fraction, polygonal number,order,rank.*

Notations:

1. $\langle a_0, a_1, a_2, a_3, \dots, a_n \rangle$– continued fraction expansion

2.$P(m,n)$ –polygonal number of order $'m'$ and rank $'n'$

AMS Classification:11A55

1 . Introduction

Number theory is an inner most preferred area for teachers and learners of mathematics. It is considered as the purest part of mathematics. Number theory presents a clear insight into the study and nature of numbers and their properties. It represents a general theory pertaining to the notion of number and its generalizations.Continued fraction appear both expectedly and unexpectedly in various areas of mathematics. The Dutch mathematician interest to build a mechanical planetarium was the motivation for his work. This field of mathematics created an interest among the great mathematicians such as Euler, Jacobi, Cauchy, Gauss and many others. Continued fractions are used in the applications of contemporary mathematics who still continue to develop the theory. A.J Van der Poortenis, the Australian mathematician is the most prominent and sufficient among them.Continued fractions may be discovered accidentally, then through examples of how rational fractions are expanded into continued fractions. Still these fractions remain more useful as they provide deep insight into many mathematical problems, particularly into the nature of numbers and remains an active subject of investigation among the great mathematicians of 17th and 18th centuries. Continued fractions are used to provide better rational approximations to irrational numbers. It also provides an introduction to the discussion of idea of limits.

1.1 Continued Fraction:

An expression of the form

$$\frac{x}{y} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}}$$

wherea_i ,b_i are real or complex numbers is called a continued fraction.

ISSN: 2005-4238 IJAST

1.2 The Continued Fraction algorithm

Continued fraction algorithm can be used to obtain the continued fraction expansion of any real number. Let y be any real number. Choose $y_0 = y$ andlet $a_0 = y_0$,

Define $y_i = \frac{1}{y_0 - [y_0]}$ and let $a_1 = [y_1]$

$y_2 = \frac{1}{y_1 - [y_1]}$, which implies that $a_2 = [y_2]$, ….., $y_k = \frac{1}{y_{k-1} - [y_{k-1}]}$, which implies that $a_k = [y_k]$, ….., This technique is extend infinitely or to some finite stage till and $y_i \in N$, exists such that $a_i = [y_i]$.

1.3Theorem :

If $'m'$ and $'n'$ are the order and rank of the polygonal number $P(m,n)$ then the continued fraction of $\frac{P(m,n)}{P(m+1,n)}$ is given by $\langle 0, 1, m-2, \frac{n-1}{2} \rangle$.

Proof:

$$\frac{P(m,n)}{P(m+1,n)} = \frac{(n^2(m-2) - n(m-4))/2}{(n^2(m+1-2) - n(m+1-4))/2}$$

$$= \frac{n(m-2) - (m-4)}{n(m-1) - (m-3)}$$

$$= 0 + \cfrac{1}{\cfrac{nm-n-m+3}{nm-2n-m+4}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{n-1}{nm-2n-m+4}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{nm-2n-m+4}{n-1}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{m-2+\cfrac{2}{n-1}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{m-2+\cfrac{1}{\cfrac{n-1}{2}}}}$$

$$\frac{P(m,n)}{P(m+1,n)} = \langle 0, 1, m-2, \frac{n-1}{2} \rangle$$

1.3 Illustration

$$\frac{P(4,n)}{P(5,n)} = \langle 0, 1, 2, \frac{n-1}{2} \rangle$$

ISSN: 2005-4238 IJAST

$$\frac{P(5,n)}{P(6,n)} = \langle 0,1,3,\frac{n-1}{2}\rangle$$

1.4Theorem

$$\frac{P(m,n)}{P(m,n+1)} = \begin{cases} \langle 0,1,\frac{n-1}{2},\frac{2mn-4n+2}{n+1}\rangle \ if\ n\ is\ odd \\ \langle 0,1,\frac{n-2}{2},1,3,\frac{n(m-3)}{2n(4-m)+2}\rangle \ if\ n\ is\ even \end{cases}$$

where m and n represents the order and rank of $P(m,n)$.

Proof

$$\frac{P(m,n)}{P(m,n+1)} = \frac{(n^2(m-2)-n(m-4))/2}{((n+1)^2(m-2)-(n+1)(m-4))/2}$$

$$= \frac{n^2(m-2)-n(m-4)}{(n^2+2n+1)(m-2)-(n+1)(m-4)}$$

$$= \frac{n^2m-2n^2-nm+4n}{n^2m+2nm+m-2n^2-2-4n-nm+4n-m+4}$$

$$= \frac{n^2m-2n^2-nm+4n}{n^2m-2n^2+nm+2}$$

$$\frac{P(m,n)}{P(m,n+1)} = 0+\frac{1}{\frac{n^2m-2n^2+nm+2}{n^2m-2n^2-nm+4n}}$$

Case (i) n **is odd**

$$\frac{P(m,n)}{P(m,n+1)} = 0+\frac{1}{1+\frac{2nm-4n+2}{n^2m-2n^2-nm+4n}}$$

$$= 0+\frac{1}{1+\frac{1}{\frac{n^2m-2n^2-nm+4n}{2nm-4n+2}}}$$

$$= 0+\frac{1}{1+\frac{1}{\left(\frac{n-1}{2}\right)+\frac{n+1}{2nm-4n+2}}}$$

$$= 0+\frac{1}{1+\frac{1}{\left(\frac{n-1}{2}\right)+\frac{1}{\frac{2nm-4n+2}{n+1}}}}$$

Take $n=2k+1, k=0,1,2,3,..$

$$\frac{P(m,n)}{P(m,n+1)} = 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{2m(2k+1) - 4(2k+1) + 2}{2k+1+1}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{4mk + 2m - 8k - 4 + 2}{2k+2}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{4mk + 2m - 8k - 2}{2k+2}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{2km + m - 4k - 1}{k+1}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{m(2k+1) - 4k - 1}{k+1}}}}$$

$$= 0$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{mn - 2n + 2 - 1}{\cfrac{n+1}{2}}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{2(mn - 2n + 1)}{n+1}}}}$$

ISSN: 2005-4238 IJAST

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-1}{2}\right) + \cfrac{1}{\cfrac{2mn - 4n + 2}{n+1}}}}$$

$$\frac{P(m,n)}{P(m,n+1)} = \langle 0, 1, \frac{n-1}{2}, \frac{2mn-4n+2}{n+1} \rangle$$

1.4.1 Illustrations

$$\frac{P(m,3)}{P(m,4)} = \langle 0, 1, 1, \frac{3m-5}{2} \rangle$$

$$\frac{P(m,5)}{P(m,6)} = \langle 0, 1, 2, \frac{5m-9}{3} \rangle$$

Case (ii) n **is even**

$$\frac{P(m,n)}{P(m,n+1)} = 0 + \cfrac{1}{1 + \cfrac{2nm - 4n + 2}{n^2 m - 2n^2 - nm + 4n}}$$

$$\frac{P(m,n)}{P(m,n+1)} = 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{n^2 m - 2n^2 - nm + 4n}{2nm - 4n + 2}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{nm - n + 2}{2nm - 4n + 2}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{\cfrac{2nm - 4n + 2}{nm - n + 2}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{\cfrac{nm - 3n}{nm - n + 2}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{1 + \cfrac{1}{\cfrac{nm - n + 2}{nm - 3n}}}}}$$

ISSN: 2005-4238 IJAST

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{8n - 2nm + 2}{nm - 3n}}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{2n(4-m) + 2}{n(m-3)}}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{n-2}{2}\right) + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{\cfrac{n(m-3)}{2n(4-m) + 2}}}}}}$$

$$\frac{P(m,n)}{P(m,n+1)} = \left\langle 0, 1, \frac{n-2}{2}, 1, 3, \frac{n(m-3)}{2n(4-m)+2} \right\rangle$$

1.4.2 Illustrations

$$\frac{P(m,4)}{P(m,5)} = \left\langle 0, 1, 1, 1, 3, \frac{2m-6}{17-4m} \right\rangle$$

$$\frac{P(m,6)}{P(m,7)} = \left\langle 0, 1, 2, 1, 3, \frac{3m-9}{25-6m} \right\rangle$$

Conclusion

In this paper, continued fractions expression of ratios of polygonal numbers with consecutive orders and ranks are analysed. Similarly, the other kinds of ratios of polygonal numbers may be studied in detail. The study can also be extended to higher dimensional figurate numbers.

References

1. P.Balamurugan,A.Gnanam,"Pattern classification of continued fractions with triangular number as base", International journal of pure and applied mathematics, Vol 119,No 15, 3413-3418, 2018.
2. A. Gnanam, S.Krithika, " Ratios of Polygonal Numbers as Continued Fractions",International journal of Engineering Science, Advanced Computing and Bio-Technology, Vol.8,No.3, pp 143-155, July-September 2017.
3. Ivan Niven, Herbert S.Zukerman, Hugh L. Montgomery," An introduction to theory of Numbers", 5th edition, John Wiley and Sons 1991.
4. "An introductory course in Elementary Number theory" by Wissam Raji,May9,2013.
5. "An alternative approach to analytic Number Theory" by Andrew Granville,2014.
6. Adams, William; Shanks,Daniel, "Mathematics of computation", 39 (159), pp 255 – 300,MR 0658231.

ISSN: 2005-4238 IJAST

7. Nathanson , M.B, " Sums of polygonal numbers ", In analytic Number Theory and Diophantine Problem: Proceedings of a *Conference at Oklahoma State University*, 1984 (Ed. A. Adolphsonetal). M. ABoston, Birkhauser, pp. 305 – 316, 1987.
8. Rosen. K.H. "Elementary Number theory and its Applications", Fourth edition, Addison –Wesley, Reading, Massachusets, 2000.
9. ShaileshShirali, "A primer on Number sequences, Mathematics student ",Vol. 45, No.2.pp. 63 – 73, 1997.
10. Telang, S.G."Number Theory",*Tata Mcgraw Hill publishing Company, Newyork,1996.*
11. Uspensky, J.V. and M.A.Heaslet, "Elementary Number Theory ". Mcgraw Hill BookCompany, 1939.
12. Vinogradev. "An Introduction to the theory of numbers", *London and Newyork, PergamonPress, 1995.*
13. Jonathan Browein, AlfvanderPoorten, Jeffrey Shallit, WadimZudilin,"NeverendingFractions.*An Introduction to continued fractions",Cambridge UniversityPress,2014.*
14. A.Venkatachalam, P.Balamurugan,"Pattern classification of continued fractions with square number as base", journal of physics: Conference series, 2019.
15. Joseph H.Silverman"A friendly introduction to Number Theory",Fourth edition,1996.
16. Neville Robbins, " Beginning Number Theory", Second edition, Narosa Publishing House,2005.

ISSN: 2005-4238 IJAST