International Journal of Grid and Distributed Computing
Vol. 13, No. 1, (2020), pp. 309-314

A SIMPLE PROOF THAT $\sum_n \frac{\mu(n)d(n)}{n} = 0$

[1]G SUDHAAMSH MOHAN REDDY , [2]S SRINIVAS RAU, [3]B UMA
[1& 2] *Faculty of Science and Technology, Icfai Foundation for Higher Education, Hyderabad-501203, INDIA*
[3]*CTW, Military College, SECUNDRERABAD-500015, INDIA*

Abstract

A simple proof is given that $\sum_n \frac{\mu(n)d(n)}{n} = 0$ using the Prime Number Theorem.

1.INTRODUCTION

Let μ(n) denote the Moebius function and d(n) the divisor function. The question " $\sum_n \frac{\mu(n)d(n)}{n} = 0$" was posed in ([4], p 1599). W.Narkiewicz in a letter indicated that it can be proved by contour integrals. In this note we use the Prime Number Theorem in the form " $\sum_n \frac{\mu(n)}{n} = 0$ " to obtain one more proof(Prop 1).

Lemma 1.

$$\prod_p (1 - \frac{2}{p^3}) = \sum_n \frac{\mu(n)d(n)}{n^s} (\operatorname{Re} s > \sigma_a)$$

Proof. For any multiplicative function f(n) we have ([1], Theorem 11.7)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p (1 + \frac{f(p)}{p^s} + ... + \frac{f(p^k)}{p^{ks}} + ...)$$

Take f(n) = μ(n)d(n), f(p) = μ(p)d(p) = = 2 and f(pk) = μ(pk)d(pk) = 0 for

k > 2. Hence the equality.

Lemma 2.

$$\prod_p (1 - \frac{2}{p^s}) = \prod_p \left\{ \left(1 - \frac{1}{p^s}\right)^2 \right\} \prod_p (1 - \frac{1}{p^{2s}(1 - \frac{1}{p^s})^2}) (\operatorname{Re} s > 1)$$

Proof. ([3], 4.4.6) We verify the factorization

2000 Mathematics Subject Classification. 11M06.
Key words and phrases. Dirichlet series; Prime Number Theorem.

ISSN: 2005-4262 IJGDC
Copyright ⓒ2020 SERSC

$$(1-\frac{2}{x}) = (1-\frac{1}{x})^2(1-\frac{1}{x^2(1-\frac{1}{x})^2})$$

and put x = p^s, take product over all primes p and rearrange using absolute convergence for Re s>1.

$$\text{RHS} = (1-\frac{1}{x})^2(\frac{x^2(1-\frac{1}{x})^2-1}{x^2(1-\frac{1}{x})^2})$$

$$= (1-\frac{1}{x})^2(\frac{(x-1)^2-1}{(x-1)^2})$$

$$= (1-\frac{1}{x})^2(\frac{x^2-2x+1-1}{x^2(1-\frac{1}{x})^2})$$

$$= (1-\frac{1}{x})^2(\frac{x^2-2x}{x^2(1-\frac{1}{x})^2})$$

$$= \frac{x-2}{x}$$

$$= 1-\frac{2}{x}$$

= LHS

Corollary 1. For Re s > 1

$$\sum_{n=1}^{\infty}\frac{\mu(n)d(n)}{n^s} = \prod_p(1-\frac{2}{p^s}) = (\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s})^2\prod_p(1-\frac{1}{p^{2s}(1-\frac{1}{p^s})^2})$$

where the last product converges uniformly for Re s > 1.

Proof. We use the product formula ([1], p 231) for Re s > 1

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \prod_p(1-\frac{1}{p^s})$$

and the convergence condition for infinite products:

$\prod_n(1-\alpha_n), 0 \le \alpha_n < 1$ converges iff $\sum_n \alpha_n$ converges.

ISSN: 2005-4262 IJGDC
Copyright ⓒ2020 SERSC

Remark 1. Recall ([1], p 97) that the Prime Number Theorem is equivalent to

$$\sum_n \frac{\mu(n)}{n} = 0$$

Lemma 3. The Cauchy product

$$(\sum_{n=1}^{\infty} \frac{\mu(n)}{n})^2 = 0$$

Proof. Write $a_n = \dfrac{\mu(n)}{n}$ and the polynomial product.

$(\sum_{m=1}^{l} a_m x^m)(\sum_{n=1}^{k} a_n x^n) = p_t(x)$ where t = k+l. Take x = 1 to get the equality

of partial sums

$$p_t(1) = s_m s_n$$

Now (Remark 1) $\lim_{m\to\infty} s_m = 0 = \lim_{n\to\infty} s_n$ and indeed the double sequence

$\{ s_m s_n \}$ converges to 0 as m, n → ∞. Hence

$$\lim_{t\to\infty} p_t(1) = 0$$

i.e., the Cauchy product converges to 0.

Proposition 1.

$$\sum_n \frac{\mu(n)d(n)}{n} = 0$$

Proof. By Cor 1 with s = 1 we have the formal Cauchy product (equality of partial sums at s = 1, to be checked for convergence)

$$\sum_n \frac{\mu(n)d(n)}{n} = \left(\sum_n \frac{\mu(n)}{n}\right)^2 \left(\sum_n^{\infty} \frac{b_n}{n}\right)$$

$\sum_n^{\infty} \frac{b_n}{n}$ converges absolutely to a positive limit L and $\left(\sum_n \frac{\mu(n)}{n}\right)^2 = 0$ by Lemma 3.

So by Mertens' Criterion (if one of the series is absolutely convergent and the other is convergent then the Cauchy product converges to the product of the limits) the

ISSN: 2005-4262 IJGDC
Copyright ⓒ2020 SERSC

Cauchy product converges to (0)(L) = 0 as claimed.

Lemma 4. *([2], Theorem 9.6.2)* If the real series $\sum_{n=1}^{\infty} u_n$ converges to a limit L

then the partial sums $s_n = 0(n)$

Proof. Let

$$\begin{aligned}\sigma_n &= \frac{s_1 + s_2 + \ldots + s_n}{n} \\ s_n &= u_1 + u_2 + \ldots + u_n \\ &= n\sigma_n - (n-1)\sigma_{n-1} \\ \therefore \frac{s_n}{n} &= \sigma_n - \frac{(n-1)}{n}\sigma_{n-1}\end{aligned}$$

But $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} s_n = L$ since

$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$

and $L = lin\inf s_n = \limsup s_n = \lim s_n$.

Hence

$$\begin{aligned}\lim_{n\to\infty} \frac{s_n}{n} &= \lim_{n\to\infty} \sigma_n - (\lim_{n\to\infty} \frac{(n-1)}{n})(\lim_{n\to\infty} \sigma_{n-1}) \\ &= L - (1)(L) \\ &= 0\end{aligned}$$

i.e., $s_n = 0(n)$.

Corollary 2.

$$\sum_{n \le x} \mu(n) d(n) = 0(x)$$

Proof. We apply ([2], Theorem 9.6.3) let

$$\begin{aligned}t_n &= u_1 + 2u_2 + \ldots + nu_n \\ &= (n+1)s_n - n\sigma_n\end{aligned}$$

Hence

ISSN: 2005-4262 IJGDC
Copyright ⓒ2020 SERSC

$$\lim_{n\to\infty}\frac{t_n}{n}=\lim_{n\to\infty}\frac{(n+1)}{n}\frac{s_n}{n}-\lim_{n\to\infty}\sigma_n$$
$$=(1)(L)-(L)$$
$$=0$$

Take $u_n=\frac{\mu(n)d(n)}{n}$ to obtain $\sum_{n\le x}\mu(n)d(n)=0(x)$.

Acknowledgements

This work is supported by Department of Science and Technology Research Project(India) SR/S4/MS: 834/13 and the support is gratefully acknowledged.

REFERENCES

[1] Tom Apostol, *Introduction to Analytic Number Theory*, Springer 1976

[2] J C Burkill and H Burkill, *A Second Course in Analysis*, Cambridge University Press 1970

[3] R Ram Murty, *Problems in Analytic Number Theory*, Springer 1999

[4] S Srinivas Rau and B Uma, *Square-free ideals and an assertion of Ramanujan*, Indian J Pure Appl Math 33(10) 1595-1600, October 2000

[5] G. Sudhaamsh Mohan Reddy, S. Srinivas Rau, B.Uma, *A Direct proof of convergence of Euler product for $L_d(1)$,* International Journal of Advanced Science and Technology, Vol. 28, No. 16, (2019), pp. 1308-1311.

[6] G.Sudhaamsh Mohan Reddy, S.Srinivas Rau, B.Uma, *Kronecker's Lemma and a Converse with applications* ,International Journal of Advanced Science and Technology Vol. 28, No. 16, (2019), pp. 1312-1314

[7] G. Sudhaamsh Mohan Reddy, S. Srinivas Rau, B. Uma, *Norms of square free Ideals in Quadratic Fields*, International Journal of Control and Automation, Vol. 12, No. 6, (2019), pp.405-407.

[8] G Sudhaamsh Mohan Reddy, S Srinivas Rau and B Uma, *Applications of Tauberian Theorems to Dirichlet Series*, Lap Lambert Academic Publishing, Germany, ISBN:978-613-9-45507-2.

[9] G Sudhaamsh Mohan Reddy, SS Rau, B Uma *A remark on Hardy-Ramanujan's approximation of divisor functions*, International Journal of Pure and Applied Mathematics 118 (4), 997-999, 2018

[10] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Some Dirichlet Series and Means of Their Coefficients*, Southeast Asian Bulletin of Mathematics 40 (4), 585-591, 2016

[11] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Some arithmetic functions and their means*, International Journal of Pure and Applied Mathematics 119 (2), 369-374, 2018

[10] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *A Bertrand Postulate for a subclass of primes*, Boletim da Sociedade Paranaense de Matemtica 31 (2), 109-111, 2013

ISSN: 2005-4262 IJGDC
Copyright ⓒ2020 SERSC

[11] G Sudhaamsh Mohan Reddy, SS Rau, B Uma, *Converegence of a series leading to an analogue of Ramanujan's assertion on squarefree integers*, Boletim da Sociedade Paranaense de Matemtica 38 (2), 83-87, 2020

[12] G Sudhaamsh Mohan Reddy, S Srinivas Rau and B Uma, *An equivalent form of the Prime Number Theorem*, Sarajevo Journal of Mathematics, Vol.15 (28), No.2, (2019), 239 – 243(DOI: 10.5644/SJM.15.02.08).

[13] G Sudhaamsh Mohan Reddy, S Srinivas Rau, A note on Dirichlet series connected to $\frac{L_d(1)}{L_d(2)} - I$ (Accepted in publication).

ISSN: 2005-4262 IJGDC
Copyright ©2020 SERSC