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New subclass of bi-univalent function associated to Horadam polynomials

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Abstract
The main purpose of this article is to make use of the Horadam polynomials in order to introduce new subclass $G_{\Sigma}(\lambda, x, n)$ of the bi-univalent function Σ , we then derive coefficient inequalities on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete-Szego inequalities for functions belonging to the defined classes.

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1 Introduction:

Let A denote the class of the function f(z) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ and satisfy the normalization condition $f(0) = 0, f'(0) = 1$.

Let S denote the subclass of A consisting functions of the form (1.1) which are univalent in U. In 1983, Salagean [1] introduced the following differential operator:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = zf'(z)$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right), \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

For the functions given by equation (1.1), we can easily find that

$$D^n f(z) = z + \sum_{n=2}^{\infty} k^n a_n z^n$$

A function $f \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$) if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U \tag{1.2}$$

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and convex of order α if and only if

$$Re\left(1+\frac{zf''(z)}{f(z)}\right) > \alpha, \qquad z \in U \tag{1.3}$$

Denote these classes respectively $S^*(\alpha)$ and $K(\alpha)$.

Every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f^{-1}(f(w)) = w$, $(|w| < r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2w + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.4}$$

A function $f \in S$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let $\sum$ denote the class of bi-univalent functions in U given by (1.1). The class $\sum$ of biunivalent functions was first investigated by Lewin [8] and it was shown that $|a_2| < 1.51$. Brannan and Taha [2] also introduced certain subclasses of bi-univalent function and estimates for their initial coefficients.

Recently, Srivastava et.al.[10], Srivastava and Bansal [11], Sourabh Porwal and M.Darus [9], Frasin and Aouf [7] are also introduced and investigated the various subclasses of bi-univalent functions and obtained bound for the coefficients $|a_2|$ and $|a_3|$. Motivating with their work, we introduce new subclass of the function class P and find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in this subclass.

For two functions f and g are analytic in U, we say that the function f is subordinate to g in U, written as $f \prec g,\ z \in U$, if there exists an analytic function $w(z)$ defined on U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. Also, it is known that $f(z) \prec g(z),\ (z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

Recently, Horcum and Kocer [12] considered Horadam polynomials $h_n(x)$ are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x)\ ,(n \in \mathbb{N} \geq 2). \tag{1.5}$$

with $h_1 = a, h_2 = bx, h_3 = pbx + aq$ for some real constants a, b, p and q. The generating function of the Horadam polynomials $h_n(x)$ (see [12]) is given by

$$\Pi(x,z) = \frac{a+(b-ap)}{1-pst-qt^2} = \sum_{n=1}^{\infty} h_n(x)\, z^{n-1} \tag{1.6}$$

Note that for particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list few cases here (see [12], [13] for more details):

1. For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$
2. For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$
3. For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$
4. For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$
5. For $a = b = 1, p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$of the first kind.
6. For $a = 1, b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$of the second kind.

More details associated with these polynomials sequences in ([12]-[15]).

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2 Coefficient Bounds for the Class $G_{\Sigma}(\lambda, x, n)$

We begin by introducing the class $G_{\Sigma}(\lambda, x, n)$ of bi-univalent functions in the following definition.

Definition 2.1. A function f ∈ P of the form (1.1) belongs to the class $G_{\Sigma}(\lambda, x, n)$, $\lambda \geq 1$ and $z, w \in U$ if the following conditions are satisfied:

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1} f(z)}{z} \prec \Pi(x, z)+1-a \tag{2.1}$$

and

$$\frac{(1-\lambda)D^n f(w)+\lambda D^{n+1} f(w)}{w} \prec \Pi(x, w)+1-a. \tag{2.2}$$

Theorem 2.1. Let the function f given by (1.1) be in the class $G_{\Sigma}(\lambda, x, n)$. Then

$$|a_2| \leq \frac{|bx|\sqrt{bx}}{b^2x^2[(1-\lambda)3^n+\lambda 3^{n+1}]^2-(pbx^2+aq)[(1-\lambda)2^n+\lambda 2^{n+1}]} \tag{2.3}$$

$$|a_3| \leq \frac{2|bx|}{(1-\lambda)3^n+\lambda 3^{n+1}}+\frac{4(bx)^2}{[(1-\lambda)2^n+\lambda 2^{n+1}]^2} \tag{2.4}$$

and

$$|a_3-ta_2^2| = \begin{cases} \dfrac{|h_2(x)|}{[(1-\lambda)3^n+\lambda 3^{n+1}]}, & 0 \leq \sigma(t,x) \leq \dfrac{1}{2[(1-\lambda)3^n+\lambda 3^{n+1}]} \\ \dfrac{2|h_2(x)||\sigma(t,x)|}{[(1-\lambda)3^n+\lambda 3^{n+1}]}, & \sigma(t,x) \geq \dfrac{1}{2[(1-\lambda)3^n+\lambda 3^{n+1}]} \end{cases}$$

where $h_2 = bx, h_3 = pbx^2 + aq$ and

$$\sigma(t,x) = \frac{(1-t)[h_2(x)]^2}{2[[h_2(x)]^3[(1-\lambda)3^n+\lambda 3^{n+1}]^2-h_3(x)[(1-\lambda)2^n+\lambda 2^{n+1}]]}.$$

Proof:
Let $f \in G_{\Sigma}(\lambda, x, n)$ be given by Taylor-Maclaurin expansion (1.1). Then, there are analytic functions $\emptyset$ and φ such that $\emptyset(0) = \varphi(0) = 0$, $|\emptyset(z)| < 1$ and $|\varphi(z)| < 1$, $(z, w \in U)$.

We can write

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1} f(z)}{z} = \omega(x, \emptyset(z))+1-a \tag{2.5}$$

and

$$\frac{(1-\lambda)D^n g(w)+\lambda D^{n+1} g(w)}{w} = \omega\big(x, \varphi(w)\big)+1-a. \tag{2.6}$$

Or, equivalently,

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1} f(z)}{z} = 1+h_1(x)-a+h_2(x)\emptyset(z)+h_3(x)[\emptyset(z)]^2+\cdots \tag{2.7}$$

and

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$$\frac{(1-\lambda)D^n g(w)+\lambda D^{n+1} g(w)}{w} = 1+\mathrm{h}_1(x)-a+h_2(x)\varphi\,(\mathrm{w})+\mathrm{h}_3(x)[\varphi\,(\mathrm{w})]^2+\cdots \tag{2.8}$$

From equation (2.7) and (2.8), we obtain

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1} f(z)}{z} = 1+\mathrm{h}_1(x)+h_2(x)\,p_1(z)+[\mathrm{h}_2(x)p_2+h_3(x)p_1^2]^2 z^2+\cdots \tag{2.9}$$

and

$$\frac{(1-\lambda)D^n g(w)+\lambda D^{n+1} g(w)}{w} = 1+\mathrm{h}_1(x)+h_2(x)\,q_1(w)+[\mathrm{h}_2(x)q_2+h_3(x)q_1^2]^2 w^2+\cdots \tag{2.10}$$

It is well known that

$$|\emptyset\,(\mathrm{z})| = |p_1 z+p_2 z^2+\cdots| < 1 \qquad \text{and} \qquad |\varphi\,(\mathrm{z})| = |q_1 w+q_2 w^2+\cdots| < 1$$

then
$|p_k| < 1$ and $|q_k| < 1$, $(k \in \mathbb{N})$.
Thus, upon comparing the corresponding coefficients in equations (2.9) and (2.10), we have

$$h_2(x)p_1 = [(1-\lambda)2^n+\lambda 2^{n+1}]a_2 \tag{2.11}$$

$$h_2(x)p_2+h_3(x)p_1^2 = [(1-\lambda)3^n+\lambda 3^{n+1}]a_3 \tag{2.12}$$

$$-h_2(x)q_1 = [(1-\lambda)2^n+\lambda 2^{n+1}]a_2 \tag{2.13}$$

and

$$h_2(x)q_2+h_3(x)q_1^2 = [(1-\lambda)3^n+\lambda 3^{n+1}](2a_2^2-a_3). \tag{2.14}$$

From (2.11) and (2.13), we find that

$$p_1 = -q_1 \tag{2.15}$$

$$2[(1-\lambda)2^n+\lambda 2^{n+1}]^2\ \ a_2^2 = [h_2(x)]^2[p_1^2+q_1^2] \tag{2.16}$$

$$a_2^2 = \frac{[h_2(x)]^2[p_1^2+q_1^2]}{2[(1-\lambda)2^n+\lambda 2^{n+1}]^2}. \tag{2.17}$$

If we add equations (2.12) and (2.14), we get

$$2[(1-\lambda)3^n+\lambda 3^{n+1}]^2\ \ a_2^2 = h_2(x)(p_2+q_2)+h_3(x)[p_1^2+q_1^2]. \tag{2.18}$$

By substituting (2.16) in equation (2.18), we obtain

$$a_2^2 = \frac{[h_3(x)]^3[p_2+q_2]}{2[h_2(x)]^2[(1-\lambda)3^n+\lambda 3^{n+1}]^2-2h_3(x)[(1-\lambda)2^n+\lambda 2^{n+1}]}$$

which yields

$$|a_2| \le \frac{|bx|\sqrt{bx}}{b^2x^2[(1-\lambda)3^n+\lambda 3^{n+1}]^2-(pbx^2+aq)[(1-\lambda)2^n+\lambda 2^{n+1}]}.$$

By subtracting equation (2.14) from (2.12), we obtain

$$[(1-\lambda)3^n+\lambda 3^{n+1}](2a_3-2a_2^2) = h_2(x)(p_2-q_2)+h_3(x)[p_1^2-q_1^2]$$

By using equation (2.15), we get

$$a_3 = \frac{h_2(x)[p_2-q_2]}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}+a_2^2\,. \tag{2.19}$$

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Now, using equation (2.17) in (2.19), we obtain

$$a_3 = \frac{h_2(x)[p_2 - q_2]}{2[(1-\lambda)3^n + \lambda 3^{n+1}]} + \frac{[h_2(x)]^2[p_1^2 + q_1^2]}{2[(1-\lambda)2^n + \lambda 2^{n+1}]^2}$$

which yields

$$|a_3| \leq \frac{2|bx|}{(1-\lambda)3^n + \lambda 3^{n+1}} + \frac{4(bx)^2}{[(1-\lambda)2^n + \lambda 2^{n+1}]^2}.$$

From (2.21), for $t \in \mathbb{R}$, we write

$$a_3 - ta_2^2 = \frac{h_2(x)[p_2 - q_2]}{2[(1-\lambda)3^n + \lambda 3^{n+1}]} + (1-t)a_2^2. \tag{2.20}$$

By substituting (2.19) in (2.20), we have

$$a_3 - ta_2^2 = \frac{h_2(x)[p_2 - q_2]}{2[(1-\lambda)3^n + \lambda 3^{n+1}]} + (1-t)\frac{[h_3(x)]^3[p_2 + q_2]}{2[h_2(x)]^2[(1-\lambda)3^n + \lambda 3^{n+1}]^2 - 2h_3(x)[(1-\lambda)2^n + \lambda 2^{n+1}]}$$
$$= h_2(x)\left\{\left(\sigma(t,x) + \frac{1}{2[(1-\lambda)3^n + \lambda 3^{n+1}]}\right)p_2 + \left(\sigma(t,x) - \frac{1}{2[(1-\lambda)3^n + \lambda 3^{n+1}]}\right)q_2\right\}$$

where

$$\sigma(t,x) = \frac{(1-t)[h_2(x)]^2}{2[[h_2(x)]^3[(1-\lambda)3^n + \lambda 3^{n+1}]^2 - h_3(x)[(1-\lambda)2^n + \lambda 2^{n+1}]]}.$$

Hence, we conclude that

$$|a_3 - ta_2^2| = \begin{cases} \dfrac{|h_2(x)|}{[(1-\lambda)3^n + \lambda 3^{n+1}]}, & 0 \leq \sigma(t,x) \leq \dfrac{1}{2[(1-\lambda)3^n + \lambda 3^{n+1}]} \\ \dfrac{2|h_2(x)||\sigma(t,x)|}{[(1-\lambda)3^n + \lambda 3^{n+1}]}, & \sigma(t,x) \geq \dfrac{1}{2[(1-\lambda)3^n + \lambda 3^{n+1}]} \end{cases}$$

which completes the proof of Theorem (2.1).

On putting $\lambda = 1, n = 0$, we get following corollary.

Corollary 2.1.1. Let the function f given by (1.1) be in the class $G_{\Sigma}(1,x)$.

$$|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{bx^2(9b-2p) - aq}}$$
$$|a_3| \leq \frac{2}{3}|bx| + (bx)^2$$

and

$$|a_3 - ta_2^2| = \begin{cases} \dfrac{|h_2(x)|}{3}, & 0 \leq \sigma(t,x) \leq \dfrac{1}{3} \\ \dfrac{2|h_2(x)||\sigma(t,x)|}{3}, & \sigma(t,x) \geq \dfrac{1}{3} \end{cases}$$

where

$$\sigma(t,x) = \frac{(1-t)[h_2(x)]^2}{2[9[h_2(x)]^3 - 2h_3(x)]}.$$

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