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1- Generator Quasi-Cyclic Codes over $F_2[u,v,w]/< u^2, v^2, w^2, uv-vu, vw-wv, uw-wu >$

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Abstract

Minimal spanning set for 1- generator quasi-cyclic(QC)codes and 1- generator generalized quasi-cyclic (GQC)codes are over the ring $R = F_2 + uF_2 + vF_2 + uvF_2 + wF_2 + uwF_2 + vwF_2 + uvwF_2$ are derived in this paper. Lower bound for the minimum distance of these codes over ring R are also derived.

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1 Introduction

Cyclic codes have a prominent place in algebraic coding theory because of their rich algebraic structure and good parameters. Cyclic codes over finite chain rings were discussed in [1, 2, 3, 9, 13, 16, 18, 21]. 1- generator quasi cyclic codes over $F_{p^m} + uF_{p^m} + \ldots + u^{s-1}F_{p^m}$ where m, s are positive integers such that $s \geq 2$ and $u^s = 0$ were obtained in [10]. In [15] QC codes over Z_q were derived and QC codes over Z_4 with some new binary codes were studied in [16]. 1- generater QC codes over $F_2 + uF_2$, where $u^2 = 0$, were derived in [18] and over the ring $F_p + uF_p$ where $u^2 = u$ in [19]. Cyclic codes over the ring $F_p[u,v,w]/< u^2, v^2, w^2, uv - vu, vw - wv, uw - wu >$ were discussed in [14]. There are many well known reasons to work on QC codes as the QC codes over finite fields are closely related to convolutional codes [5] and meet a modified version of Gilbert-Varshamov bound [12].

The structure of 1-generator quasi-cyclic codes was investigated in [7, 9, 20] and a polynomial approach is presented in [11]. The structure of generalized quasi-cyclic codes (GQC) over finite fields, was discussed with their generators and a BCH type bound for these codes in [7, 9, 18]. Further this study has been extended in [3, 4]. In [2] QC codes over the ring $F_2 + uF_2 + vF_2 + uvF_2$, and some new best known binary linear codes as gray images of QC codes are obtained.

Motivated by the studies in [17], we obtained 1- generator QC codes and GQC codes over the ring $F_2[u,v,w]/< u^2, v^2, w^2, uv - vu, vw - wv, uw - wu >$.

2 Preliminaries

Throughout this paper R denotes the ring $F_2 + uF_2 + vF_2 + uvF_2 + wF_2 + uwF_2 + vwF_2 + uvwF_2$ where $u^2 = v^2 = w^2 = 0$, uv =vu, vw= wv, uw=wu.

A linear code $\mathcal{C}$ over R of length n is an R-submodule of R^n. Lee Weight of element $a+ub+vc+uvd+we+uwf+vwg+uvwh$ of R where a, b, c, d, e, f, g, h are elements of F_2 is $W_H(\Phi(a+ub+vc+uvd+we+uwf+vwg+uvwh)) = W_H(h, f+h, g+h, d+h, e+f+g+h, b+d+f+h, c+d+g+h, a+b+c+d+e+f+g+h)$ where W_H is the Hamming Weight and $\Phi: R \to F_2^8$ defined by $\Phi(a+ub+vc+uvd+we+uwf+vwg+uvwh) = (h, f+h, g+h, d+h, e+f+g+h, b+d+f+h, c+d+g+h, a+b+c+d+e+f+g+h)$ is a Gray map.
Lee Weight $W_L(c)$ of n- tuple in R^n is obtained as the rational sum of Lee weights of its components. Minimum Lee distance of a linear code $\mathcal{C}$ over R is $d_L(c)$ = min $\{ W_L(c) : 0 \neq c \in \mathcal{C} \}$. Similarly, is defined the minimum Hamming Weight $d_H(\mathcal{C})$ of $\mathcal{C}$ over R as $d_H(\mathcal{C}) = min\{d_H(c): 0 \neq c \in \mathcal{C}\}$. The Gray map naturally extend to R^n as distance preserving isometry Φ: (R^n, Lee Weight) $\to$ (F_2^8, Hamming Weight) as $\Phi(\alpha_1, \alpha_2, \dots, \alpha_n) \to (\Phi(\alpha_1), \Phi(\alpha_2), \dots, \Phi(\alpha_n))$ where $\alpha_i \in R$ for all $1 \leq i \leq n$.

Theorem 2.1 *Let $\mathcal{C}$ is a linear code of size 2^k, minimum distance d and length n over R, then $\Phi(\mathcal{C})$ is a binary linear code with parameters [8n,k,d].*

Complete ideal structure of a cyclic code $\mathcal{C}$ over R of length n is as follows:

Theorem 2.2 *[14] Let $\mathcal{C}$ be a cyclic code of length n over the ring R. Then*
(i)If n is even, then $\mathcal{C} = < \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8 >$ where
$\tau_1 = \theta_1(x) + u\xi_{1,2}(x) + v\xi_{1,3}(x) + uv\xi_{1,4}(x) + w\xi_{1,5}(x) + uw\xi_{1,6}(x) + vw\xi_{1,7}(x) + uvw\xi_{1,8}(x)$
$\tau_2 = u\theta_2(x) + v\xi_{2,3}(x) + uv\xi_{2,4}(x) + w\xi_{2,5}(x) + uw\xi_{2,6}(x) + vw\xi_{2,7}(x) + uvw\xi_{2,8}(x)$
$\tau_3 = v\theta_3(x) + uv\xi_{3,4}(x) + w\xi_{3,5}(x) + uw\xi_{3,6}(x) + vw\xi_{3,7}(x) + uvw\xi_{3,8}(x)$
$\tau_4 = uv\theta_4(x) + w\xi_{4,5}(x) + uw\xi_{4,6}(x) + vw\xi_{4,7}(x) + uvw\xi_{4,8}(x)$
$\tau_5 = w\theta_5(x) + uw\xi_{5,6}(x) + vw\xi_{5,7}(x) + uvw\xi_{5,8}(x)$
$\tau_6 = uw\theta_6(x) + vw\xi_{6,7}(x) + uvw\xi_{6,8}(x)$
$\tau_7 = vw\theta_7(x) + uvw\xi_{7,8}(x)$
$\tau_8 = uvw\theta_8(x)$
and$\theta_4(x)|\theta_2(x)|\theta_1(x)|x^n - 1$, $\theta_4(x)|\theta_3(x)|\theta_1(x)|x^n - 1, \theta_8(x)|\theta_6(x)|\theta_5(x)|x^n - 1$, $\theta_8(x)|\theta_7(x)|\theta_5(x)|\theta_1(x)$, $\theta_6(x)|\theta_2(x)$ and $\theta_7(x)|\theta_3(x)$ over F_2.

(ii) for odd n, $\mathcal{C} =< \theta_1(x) + u\theta_2(x), v\theta_3(x) + uv\theta_4(x), w\theta_5(x) + uw\theta_6(x), wv\theta_7(x) + uvw\theta_8(x) >$
Where$\theta_4(x)|\theta_2(x)|\theta_1(x)|x^n - 1$, $\theta_4(x)|\theta_3(x)|\theta_1(x)|x^n - 1, \theta_8(x)|\theta_6(x)|\theta_5(x)|x^n - 1$, $\theta_8(x)|\theta_7(x)|\theta_5(x)|\theta_1(x)$, $\theta_6(x)|\theta_2(x)$ and $\theta_7(x)|\theta_3(x)$ over F_2.

(iii) Also $\mathcal{C}$ is free cyclic code if and only if $\mathcal{C} = < \tau_1 >$ and $\tau_1|x^n - 1$ in $R[x]$.
Theorem 2.3 *[2] If $\mathcal{C} = < g(x) >$ is cyclic code over R of length n such that $g(x)|x^n - 1$ in $R[x]/< x^m - 1 >$ and f(x) $\in R$ is co-prime to $g(x)$ then $\mathcal{C} = < (f(x).g(x)) >$.*

Theorem 2.4 *[22] If $\mathcal{C}$ is cyclic code over a finite field $GF(q)$ of length n and s be the number of consecutive power of the n^{th} roots of unity that are zeros of $g(x)$ then $d_H(\mathcal{C}) \geq s + 1$.*

3 1-Generator Quasi-cyclic Codes Over R

For positive integer ℓ, the linear code $\mathcal{C}$, which is invariant under ℓ cyclic shifts, is called an ℓ - quasi cyclic code. Such a smallest ℓ is known as index of $\mathcal{C}$. Under polynomial notations, $\mathcal{C}$ is ℓ-quasi cyclic code of length $n = m\ell$ over R if and only if $\mathcal{C}$ is an $R_m[x] = R[x]/< x^m - 1 >$-submodule of $(R_m[x])^\ell$. An r-generator QC code is a R_m-submodule of R_m^ℓ with r-generators. Thus, 1-generator QC code $\mathcal{C}$ over R generated by $F(x) \in (R_m)^\ell$ is $\{\xi(x)F(x): \xi(x) \in R_m\}$.

Theorem 3.1*If $\mathcal{C}$ is a 1-generator ℓ-QC code over the ring R of length $n = m\ell$ generated by $P(x) = (P_1(x), P_2(x), \dots, P_\ell(x)) \in (R_m[x])^\ell$. For each i, $1 \le i \le \ell$, $P_i(x)$ belongs to a cyclic code $\mathcal{C}_i$ over R of length m and is of the form $\theta_{1i}(\xi_1(x) + u\xi_2(x)) + \theta_{2i}(v\xi_3(x) + uv\xi_4(x)) + \theta_{3i}(w\xi_5(x) + uw\xi_6(x)) + \theta_{4i}(wv\xi_7(x) + uvw\xi_8(x))$ when m is odd and of the form $\theta_{1i}\tau_1 + \theta_{2i}\varsigma_2 + \theta_{3i}\tau_3 + \theta_{4i}\tau_4 + \theta_{5i}\tau_5 + \theta_{6i}\tau_6 + \theta_{7i}\tau_7 + \theta_{8i}\tau_8$ when m is even, for some $\theta_{1i}, \theta_{2i}, \dots, \theta_{8i} \in R_m[x]$ and $\xi_1, \xi_2, \dots, \xi_8$ are defined in Theorem 2.2.*

Proof. For $1 \le i \le \ell$, Assume projection map $\prod_i (P_1(x), P_2(x), \dots, P_\ell(x)) \longrightarrow P_i(x)$. If $\mathcal{C} =< (P_1(x), P_2(x), \dots, P_l(x)) >$ is an ℓ-QC code of length $n = m\ell$, then due to Theorem 2.2 $\prod_i (\mathcal{C})$ is a cyclic code over R of length m.

Consider $\mathcal{C}$ to be a 1-generator ℓ-QC code over R of length $n = m\ell$ generated by $X_1 = (\xi_1 g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1, \dots, \xi_\ell g_l + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell)$ where $\xi_i, g_i, k_i, p_i, r_i, s_i, y_i, z_i$ in $F_2[x]$, g_i divides $x^m - 1$ for $1 \le i \le \ell$. Assume that i_0 be selected such that $1 \le i_0 \le \ell$ and $(\xi_{i_0} g_{i_0} + uk_{i_0} + vp_{i_0} + uvq_{i_0} + wr_{i_0} + uws_{i_0} + vwy_{i_0} + uvwz_{i_0})$ does not devides $x^m - 1$. Further assume

$g = gcd(\xi_1 g_1, \xi_2 g_2, \dots, \xi_l g_l, x^m - 1)$ with $g\mu_1 = x^m - 1$, $\deg(\mu_1) = t_1$,
$\rho_1 = gcd(k_1\mu_1, k_2\mu_1, \dots, k_l\mu_1, x^m - 1)$ with $\rho_1\mu_2 = x^m - 1$, $\deg(\mu_2) = t_2$,
$\rho_2 = gcd(p_1\mu_1\mu_2, p_2\mu_1\mu_2, \dots, p_l\mu_1\mu_2, x^m - 1)$ with $\rho_2\mu_3 = x^m - 1$, $\deg(\mu_3) = t_3$,
$\rho_3 = gcd(q_1\mu_1\mu_2\mu_3, q_2\mu_1\mu_2\mu_3, \dots, q_l\mu_1\mu_2\mu_3, x^m - 1)$ with $\rho_3\mu_4 = x^m - 1$, $\deg(\mu_4) = t_4$,
$\rho_4 = gcd(r_1\mu_1\mu_2\mu_3\mu_4, r_2\mu_1\mu_2\mu_3\mu_4, \dots, r_l\mu_1\mu_2\mu_3\mu_4, x^m - 1)$ with $\rho_4\mu_5 = x^m - 1$, $\deg(\mu_5) = t_5$,
$\rho_5 = gcd(s_1\mu_1\mu_2\mu_3\mu_4\mu_5, s_2\mu_1\mu_2\mu_3\mu_4\mu_5, \dots, s_l\mu_1\mu_2\mu_3\mu_4\mu_5, x^m - 1)$ with $\rho_5\mu_6 = x^m - 1$,$\deg(\mu_6) = t_6$,
$\rho_6 = \gcd(y_1\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6, y_2\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6, \dots, y_l\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6, x^m - 1)$with $\rho_6\mu_7 = x^m - 1$, $\deg(\mu_7) = t_7$,
$\rho_7 = \gcd(z_1\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7, z_2\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7, \dots, z_l\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7, x^m - 1)$ with $\rho_7\mu_8 = x^m - 1$, $\deg(\mu_8) = t_8$
and $X_i = \mu_{i-1}X_{i-1}$ for $2 \le i \le 8$.

Theorem 3.2*The minimal spanning set for 1-generator ℓ-QC codes $\mathcal{C}$ is given by*

$$Z_1 = \{X_1, xX_1, \dots, x^{t_1-1}X_1\} \quad Z_2 = \{X_2, xX_2, \dots, x^{t_2-1}X_2\}$$
$$Z_3 = \{X_3, xX_3, \dots, x^{t_3-1}X_3\} \quad Z_4 = \{X_4, xX_4, \dots, x^{t_4-1}X_4\}$$
$$Z_5 = \{X_5, xX_5, \dots, x^{t_5-1}X_5\} \quad Z_6 = \{X_6, xX_6, \dots, x^{t_6-1}X_6\}$$
$$Z_7 = \{X_7, xX_7, \dots, x^{t_7-1}X_7\} \quad Z_8 = \{X_8, xX_8, \dots, x^{t_8-1}X_8\}$$

such that $|\mathcal{C}| = 2^{8t_1}8^{t_2}8^{t_3}8^{t_4}4^{t_5}4^{t_6}4^{t_7}2^{t_8}$.

Proof. Let $c(x) \in \mathcal{C}$, then $c(x) = \xi(x)X_1$ for some $\xi(x) \in R_m[x]$. Due to division algorithm, there exists $Q_1, R_1 \in R_m[x]$ such that $\xi(x) = Q_1\mu_1 + R_1$, where $R_1 = 0$ or $\deg(R_1) < t_1$. Since $g = gcd(\xi_1 g_1, \xi_2 g_2, \dots, \xi_l g_l, x^m - 1)$ so for each $1 \le i \le \ell$, there exists $b_{i1} \in R_m[x]$ such that $\xi_i g_i = b_{i1} g$ and therefore $\xi_i g_i \mu_1 = b_{i1} g \mu_1 = 0$. Hence

$c(x) = \xi(x)X_1 = Q_1\mu_1 X_1 + R_1 X_1 = Q_1 X_2 + R_1 X_1$ (1)

Since $\deg(R_1) < t_1$, so $R_1 X_1 \in \text{span}(Z_1)$.

Again apply division algorithm, to obtain $Q_2, R_2 \in R_m[x]$ such that $Q_1(x) = Q_2\mu_2 + R_2$, where $R_2 = 0$ or $\deg(R_2) < t_2$. Since $\rho_1 = \gcd(k_1\mu_1, k_2\mu_1, \dots, k_l\mu_1, x^m - 1)$ so for each $1 \le i \le \ell$, some $b_{i2} \in R_m[x]$ such that $k_i\mu_1 = b_{i2}\rho_1$ and therefore $k_i\mu_1\mu_2 = b_{i2}\rho_1\mu_2 = 0$. Then (1) becomes,

$c(x) = Q_1 X_2 + R_1 X_1 = Q_2 X_3 + R_2 X_2 + R_1 X_1$ (2)

Also $\deg(R_2) < t_2$, therefore $R_2 X_2 \in \text{span}(Z_2)$.

Following the same procedure, to obtain $Q_3, R_3 \in R_m[x]$ such that $Q_2(x) = Q_3\mu_3 + R_3$, where $R_3 = 0$ or $\deg(R_3) < t_3$. Since $\rho_2 = \gcd(p_1\mu_1\mu_2, p_2\mu_1\mu_2, \dots, p_l\mu_1\mu_2, x^m - 1)$ thus again an element $b_{i3} \in R_m[x]$ such that $p_i\mu_1\mu_2 = b_{i3}\rho_2$ for each $1 \le i \le \ell$ and hence $p_i\mu_1\mu_2\mu_3 = b_{i3}\rho_2\mu_3 = 0$. Using (2), to obtain

$c(x) = Q_2 X_3 + R_2 X_2 + R_1 X_1 = Q_3 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$(3)

Here $\deg(R_3) < t_3$ thus $R_3 X_3 \in \text{span}(Z_3)$.

Continuing the similar reasoning, to obtain $Q_4, R_4 \in R_m[x]$ such that $Q_3(x) = Q_4\mu_3 + R_4$, where $R_4 = 0$ or $\deg(R_4) < t_4$ and (3) is expressed as

$c(x) = Q_3 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1 = Q_4 X_5 + Q_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$ (4)

where $\deg(R_4) < t_4$ and $R_4 X_4 \in \text{span}(Z_4)$.

Then $Q_5, R_5 \in R_m[x]$ such that $Q_4(x) = Q_5\mu_5 + R_5$, where $R_5 = 0$ or $\deg(R_5) < t_6$ and by (4), $c(x) = Q_4 X_5 + Q_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$

$= Q_5 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$(5)

where $\deg(R_5) < t_5$ and $R_5 X_5 \in \text{span}(Z_5)$.

Further $Q_6, R_6 \in R_m[x]$ such that $Q_5(x) = Q_6\mu_6 + R_6$, where $R_6 = 0$ or $\deg(R_6) < t_6$ and by (5)

$c(x) = Q_5 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$

$= Q_6 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$ (6)

where $\deg(R_6) < t_6$ and $R_6 X_6 \in \text{span}(Z_6)$.

Again $Q_7, R_7 \in R_m[x]$ such that $Q_6(x) = Q_7\mu_7 + R_7$, where $R_7 = 0$ or $\deg(R_7) < t_7$ and by (6) $c(x) = Q_6 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$

$= Q_7 X_8 + R_7 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$ (7)

where $\deg(R_7) < t_7$ and $R_7 X_7 \in \text{span}(Z_7)$.

Finally $Q_8, R_8 \in R_m[x]$ such that $Q_7(x) = Q_8\mu_8 + R_8$, where $R_8 = 0$ or $\deg(R_8) < t_8$ and by (7) $c(x) = Q_7 X_8 + R_7 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$

$= R_8 X_8 + R_7 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$

Also, $\deg(R_8) < t_8$ hence $R_8 X_8 \in \text{span}(Z_8)$.

Thus we have $c(x) = R_8 X_8 + R_7 X_7 + R_6 X_6 + R_5 X_5 + R_4 X_4 + R_3 X_3 + R_2 X_2 + R_1 X_1$ in the span of $Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_6 \cup Z_7 \cup Z_8$. Hence $Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_6 \cup Z_7 \cup Z_8$ spans $\mathcal{C}$.

Now assume that $e(x) = (e_1(x), e_2(x), \dots, e_\ell(x)) \in \text{span}(Z_1) \cap \text{span}(Z_2)$. Also suppose that there exists i such that $\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i$ does not divide $x^m - 1$. Since $e(x) \in$

span(Z_2), so $e_i(x) = (uk_i\mu + vp_i\mu + uvq_i\mu + wr_i\mu + uws_i\mu + vwy_i\mu + uvez_i\mu)M_1$ where $M_1 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{t_2-1}x^{t_2-1} \in R_m[x]$. Thus, we obtain $uvwe_i(x) = 0$.

Again $e(x) \in$ span(Z_1), thus $e_i(x) = (\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)M_2$ where $M_2 = \beta_0 + \beta_1 x + \cdots ... + \beta_{t_1-1}x^{t_1-1} \in R_m[x]$. Since $uvwe_i(x) = 0$ so $uvw(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)M_2 = uvwf_i g_i M_2 = 0$.Therefore each β_j is non-unit element of R. Moreover

$e_i(x) = (uk_i\mu_1 + vp_i\mu_1 + uvq_i\mu_1 + wr_i\mu_1 + uws_i\mu_1 + vwy_i\mu_1 + uvez_i\mu_1)M_1 = (\xi_i g_i + uk_i + vp_i + uvqi+wri+uwsi+vwyi+uvwziM2$,so $uv\xi igiM2+uvwriM2=uvwri\mu 1M1$,

$uw\xi_i g_i M_2 + uvwp_i M_2 = uvwp_i\mu_1 M_1$ and $vw\xi_i g_i M_2 + uvwk_i M_2 = uvwk_i\mu_1 M_1$ and hence at least one α_j must be a unit in R.

Further, $(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)M_2 = (uk_i\mu_1 + vp_i\mu_1 + uvq_i\mu_1 + wr_i\mu_1 + uws_i\mu_1 + vwy_i\mu_1 + uvez_i\mu_1)M_1$ which implies $(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)(M_2 + \mu_1 M_1) = 0$.

However, $(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ does not divide $x^m - 1$ and $\deg(M_2 + \mu_1 M_1) < m$, therefore $M_2 + \mu_1 M_1 = 0$ which is possible only when M_2 and M_1 both are zero elements of $R_m[x]$. Therefore span(Z_1) $\cap$ span(Z_2) = 0. On similar lines it can be concluded that span(Z_i) $\cap$ span (Z_j) = 0 for $i \neq j$ and $1 \leq i, j \leq 8$. Hence Z is linearly independent and so forms a minnimal generating set for $\mathcal{C}$.

Relaxing the condition and assuming that $(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ divides $x^m - 1$ for $1 \leq i \leq \ell$, in next theorem, the minimal generating set for a free 1-generator ℓ-QC code of length $n = m\ell$ over R are represented.

Consider $\mathcal{C}$ to be a 1-generator ℓ-QC code of length $n = m\ell$ over R generated by $X = (\xi_1 g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1, \ldots, \xi_\ell g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell)$ where $\xi_i, g_i, k_i, p_i, q_i, r_i, s_i, y_i, z_i \in F_2[x]$, g_i divides $x^m - 1$ and $\xi_i(g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ divides $x^m - 1$ for $1 \leq i \leq \ell$, g+uk+vp+uvq+wr+uws+vwy+uvwz = gcd (X,$x^m - 1$) over R with deg(g) = t and g(x)h(x) = $x^m - 1$.

Theorem 3.3 *$\mathcal{C}$ is a free R- module having minimal spanning set $X_1 = \{X, xX, \ldots, x^{m-t-1}X\}$ and $|C| = 2^{8(m-t-1)}$.*

Proof. Since gcd (X,$x^m - 1$) = (g+uk+vp+uvq+wr+uws+vwy+uvwz), so there exists $(h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz')$ in $R_m[x]$ such that (g+uk+vp+uvq+wr+uws+vwy+uvwz)$(h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = x^m - 1$ which further implies $(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)(h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = 0$ in $R_m[x]$. Also $\mu(g + uk + vp + uvq + wr + uws + vwy + uvwz)\mu(h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = gh = 0$ in $F_2[x]/< x^m - 1 >$.

Thus $(uk_i h + vp_i h + uvq_i h + wr_i h + uws_i h + vwy_i h + uvwz_i h) + (\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)(uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = 0$ and hence $(uk_i h + vp_i h + uvq_i h + wr_i h + uws_i h + vwy_i h + uvwz_i h) = (\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)(uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz')$ for $1 \leq i \leq \ell$.

Let $c(x) \in \mathcal{C}$, then $c(x) = \xi(x)X$ for some $\xi(x) \in R_m[x]$. Use division algorithm to obtain $Q_1, R_1 \in R_m[x]$ such that $\xi(x) = Q_1 h + R_1$ where $R_1 = 0$ or $deg R_1 < m - t$. Then

$$c(x) = \xi(x)X = (Q_1(uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') + R_1)(\xi_1 g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1, \dots, \xi_\ell g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell)$$

Hence $c(x) \in \text{span}(X_1)$. Therefore, X_1 spans $\mathcal{C}$.
Further, let there exists a non zero polynomial $e(x) \in R_m[x]$, with deg(e(x)) $< m - t$ such that $e(x)X = 0$ and therefore, $e(x)(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i) = 0$ for $1 \le i \le \ell$. Hence $\xi_i g_i \mu(e(x)) = 0$ in $F_2[x]/< x^m - 1 >$ however, g divides $\xi_i g_i$ for $1 \le i \le \ell$, so $g\mu(e) = 0$. Therefore, $x^m - 1$ divides $g(x)\mu(e(x))$ and so h divides $\mu(e(x))$. Hence deg$(e(x)) > m - t$, which contradicts the assumption. Therefore X_1 is lineary independent.

In next theorem, assume $\mathcal{C}$ to be the 1-generator ℓ-QC code of length $n = m\ell$ over R generated by $X = (\xi_1(g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1), \dots, \xi_\ell(g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell)), g_i, k_i, p_i, q_i, r_i, s_i, y_i, z_i \in F_2[x], \xi_i \in R[x]$ and g_i divides $(x^m - 1)$. Also consider $(g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ divides $x^m - 1$ and $\gcd(\xi_i, \frac{x^m - 1}{g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i}) = 1$ for $1 \le i \le \ell$.

Theorem 3.4*If g+uk+vp+uvq+wr+uws+vwy+uvwz = gcd($g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1, \dots, g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell$) with deg(g+uk+vp+uvq+wr+uws+*vwy+uvwz) = t, then $\mathcal{C}$ is free module with basis $Z = \{X, xX, \dots x^{m-t-1}X\}$ and $|C| = 2^{8(m-t)}$. Further, $d_H(\mathcal{C}) \ge \min_{i=1,2,\dots,l}\{\alpha_i + 1\}$ where α_i is the number of consecutive powers of the m^{th} roots of unity satisfying $g_i(x)$.

Proof. Let
$h'_i + uk'_i + vp'_i + uvq'_i + wr'_i + uws'_i + vwy'_i + uvwz'_i = \frac{x^m - 1}{g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i}$ and $h' + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz'$ = lcm $(h'_1 + uk'_1 + vp'_1 + uvq'_1 + wr'_1 + uws'_1 + vwy'_1 + uvwz'_1, \dots, h'_\ell + uk'_\ell + vp'_\ell + uvq'_\ell + wr'_\ell + uws'_\ell + vwy'_\ell + uvwz'_\ell)$. Then $h' + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz' = \frac{x^m - 1}{g + uk + vp + uvq + wr + uws + vwy + uvwz}$.
Let $c(x) \in \mathcal{C}$, then $c(x) = \xi(x)X$ for some $\xi(x) \in R_m[x]$, so there exists $Q_1, R_1 \in R_m[x]$ such that ξ(x) = $Q_1 h' + R_1$ where $R_1 = 0$ or $deg R_1 < deg h'$. Thus, $c(x) = \xi(x)X = (Q_1(uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') + R_1)(\xi_1(g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1), \dots, \xi_\ell(g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell))$
Hence $c(x) \in \text{span}(X_1)$. Therefore, X_1 spans $\mathcal{C}$.
Assume $\xi(x)X = 0$ for some non-zero element $R_m[x]$ with deg$(\xi(x)) < m - t$. Thus, $\xi(x)\xi_i(x)(g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i) = 0$ for $1 \le i \le \ell$ and so $x^m - 1$ divides $\xi(x)\xi_i(x)(g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ for $1 \le i \le \ell$. Since gcd$(\xi_i(x), h'_i + uk'_i + vp'_i + uvq'_i + wr'_i + uws'_i + vwy'_i + uvwz'_i) = 1$. So $h'_i + uk'_i + vp'_i + uvq'_i + wr'_i + uws'_i + vwy'_i + uvwz'_i$ divides $\xi(x)$ and hence $h' + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz'$ divides $\xi(x)$. Therefore, deg$(\xi(x)) >$ deg$(h' + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = m - t$ which contraducts the assumption. Thus, Z is linearly independent.

Further, let $c = (c_1, c_2, \dots\dots, c_l)$ be any non zero codeword of $\mathcal{C}$. Then, at least one c_j for $1 \le j \le \ell$, is different form zero. This implies that $c_j \in \prod_j (C) =< \xi_j(x)(g_j + uk_j + vp_j + uvq_j + wr_j + uws_j + vwy_j + uvwz_j) >=< g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i >$. Therefore, due to theorem 2.4, the non zero weight of $c_j \in< g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i >$ is atleast $\alpha_j + 1$. Hence the result follows.

Remark If in theorem 3.4, in place of $\mathcal{C}$ to be generated by $X = (\xi_1(g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1), \dots, \xi_\ell(g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell))$ we consider $\mathcal{C}$ to be generated by $X = (\xi_1(g + uk + vp + uvq + wr + uws + vwy + uvwz), \dots\dots, \xi_\ell(g + uk + vp + uvq + wr + uws + vwy + uvwz))$ and rest assumptions remain invariant, then following the same procedure, $\mathcal{C}$ is free module with basis $Z = \{X, xX, \dots x^{m-t-1}X\}$ and $|C| = 2^{8(m-t)}$.
Further, assume that

$$h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz' = \frac{x^m - 1}{g + uk + vp + uvq + wr + uws + vwy + uvwz}$$

and $c = (c_1, c_2, \dots\dots, c_l) \in \mathcal{C}$ be any code word. Then $c = aX$ for some $a \in R[x]$. Clearly $deg(a(x)) < m - t$. If $c_i = 0$ for some $1 \le i \le \ell$, that is, $a\xi_i(g + uk + vp + uvq + wr + uws + vwy + uvwz) = 0$ then $x^m - 1$ divides $a\xi_i(g + uk + vp + uvq + wr + uws + vwy + uvwz)$ which implies that $h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz')$ divides $a\xi_i$. Since $gcd(\xi_i, h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz') = 1$, so $h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz'$ divides a. Also, $deg(a(x)) < deg(h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz')$ and $h + uk' + vp' + uvq' + wr' + uws' + vwy' + uvwz'$ being monic in $R[x]$, therefore $\alpha = 0$ and hence $c = 0$. Thus, if c is a non-zero codeword in $\mathcal{C}$ then all of its components must be non-zero. As $\prod_i (\mathcal{C}) =< \xi_i(g + uk + vp + uvq + wr + uws + vwy + uvwz) >=< g + uk + vp + uvq + wr + uws + vwy + uvwz >$ and $d_H(\prod_i (c)) = d_H(< g >)$. So if g has $'\alpha'$ number of m^{th} roots of unity, then the hamming weight of each non-zero component will be $\ge (\alpha + 1)$ and hence $d_H(\mathcal{C}) \ge \ell(\alpha + 1)$ where α is the number of consecutive powers of the m^{th} roots of unity which satisfy $g(x)$.

41-Generator Generalized Quasi-Cyclic Code (GQC)

If $\lambda_1, \lambda_2, \dots, \lambda_\ell$ are integers such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$, $\lambda_i > 0$ and $R_i[x] = R[x]/< x^{\lambda_i} - 1 >$ for $1 \le i \le \ell$, then the cartesian product $\mathfrak{R} = R_1 \times R_2 \times \dots \times R_\ell$ is an $R[x]$-module and generalized quasi-cyclic (GQC) codes of length $(\lambda_1 + \lambda_2 + \cdots, \lambda_\ell)$ over the ring R is an $R[x]$-submodule of $\mathfrak{R}$. For $\lambda_1 = \lambda_2 = \cdots = \lambda_\ell = m$, a GQC code of length $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a quasi-cyclic code of length $n = m\ell$ and index ℓ over R. An κ-generator GQC code over R is an $R[x]$-submodule of $\mathfrak{R}$ with κ-generator.
In present section we only study 1-generator GQC code of length $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ over the ring R. A 1-generator GQC code $\mathcal{C}$ over R spanned by $P(x) = (P_1(x), P_2(x), \dots\dots, P_\ell(x))$ where $P_i[x] \in \frac{R[x]}{<x^{\lambda_i} - 1>}$ is defined by $\mathcal{C} = \{\xi(x)P(x): \xi(x) \in R[x]\}$. Next result help in obtaining minimal spanning set of 1-generator GQC over R.

Theorem 4.1*Let $\mathcal{C}$ be a 1-generator GQC code of length $(\lambda_1, \lambda_2, \dots \lambda_\ell)$ over R generated by $X_1 = (\xi_1 g_1 +$*

$uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1, \dots, \xi_\ell g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell)$ *where* $\xi_i, g_i, k_i, q_i, r_i, s_i, y_i, z_i \in Z_2[x]$, g_i *divides* $x^{\lambda_i} - 1$ *for* $1 \le i \le \ell$ *and there exists* i_0, $1 \le i_0 \le \ell$ *such that* $\xi_{i_0} g_{i_0} + uk_{i_0} + vp_{i_0} + uvq_{i_0} + wr_{i_0} + uws_{i_0} + vwy_{i_0} + uvwz_{i_0}$ *does not divides* $x^{\lambda_i} - 1$.

Let $\mu_{1i} = \frac{x^{\lambda_i}-1}{gcd\,(\xi_i g_i, x^{\lambda_i}-1)}$, $\mu = lcm(\mu_1, \mu_2, \dots, \mu\ell)$ with $\deg(\mu) = t_1$

$\mu_{2i} = \frac{x^{\lambda_i}-1}{gcd\,(k_i\mu_1, x^{\lambda_i}-1)}$, $\mu_2 = lcm(\mu_{21}, \mu_{22}, \dots, \mu_{2\ell})$ with $\deg(\mu_2) = t_2$

$\mu_{3i} = \frac{x^{\lambda_i}-1}{gcd\,(p_i\mu_1\mu_2, x^{\lambda_i}-1)}$, $\mu_3 = lcm(\mu_{31}, \mu_{32}, \dots, \mu_{3\ell})$ with $\deg(\mu_3) = t_3$

$\mu_{4i} = \frac{x^{\lambda_i}-1}{gcd\,(q_i\mu_1\mu_2\mu_3, x^{\lambda_i}-1)}$, $\mu_4 = lcm(\mu_{41}, \mu_{42}, \dots, \mu_{4l})$ with $\deg(\mu_4) = t_4$

$\mu_{5i} = \frac{x^{\lambda_i}-1}{gcd\,(r_i\mu_1\mu_2\mu_3\mu_4, x^{\lambda_i}-1)}$, $\mu_5 = lcm(\mu_{51}, \mu_{52}, \dots, \mu_{5\ell})$ with $\deg(\mu_5) = t_5$

$\mu_{6i} = \frac{x^{\lambda_i}-1}{gcd\,(s_i\mu_1\mu_2\mu_3\mu_4\mu_5, x^{\lambda_i}-1)}$, $\mu_6 = lcm(\mu_{61}, \mu_{62}, \dots, \mu_{6\ell})$ with $\deg(\mu_6) = t_6$

$\mu_{7i} = \frac{x^{\lambda_i}-1}{gcd\,(y_i\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6, x^{\lambda_i}-1)}$, $\mu_7 = lcm(\mu_{71}, \mu_{72}, \dots, \mu_{7l\ell})$ with $\deg(\mu_7) = t_7$

$\mu_{8\ell} = \frac{x^{\lambda_i}-1}{gcd\,(z_i\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7, x^{\lambda_i}-1)}$, $\mu_8 = lcm(\mu_{81}, \mu_{82}, \dots, \mu_{8l})$ with $\deg(\mu_8) = t_8$ and $X_i = \mu_{i-1}X_{i-1}$ for $2 \le i \le 8$.

Then the minimal spanning set of C is given by

$$Z_1 = \{X_1, xX_1, \dots, x^{t_1-1}X_1\} Z_2 = \{X_2, xX_2, \dots, x^{t_2-1}X_2\}$$
$$Z_3 = \{X_3, xX_3, \dots, x^{t_3-1}X_3\} Z_4 = \{X_4, xX_4, \dots, x^{t_4-1}X_4\}$$
$$Z_5 = \{X_5, xX_5, \dots, x^{t_5-1}X_5\} Z_6 = \{X_1, xX_6, \dots, x^{t_6-1}X_6\}$$
$$Z_7 = \{X_1, xX_7, \dots, x^{t_7-1}X_7\} \qquad Z_8 = \{X_8, xX_8, \dots, x^{t_8-1}X_8\}$$

Proof. Proof can be obtained on similar lines of that of theorem 3.2.

Theorem 4.2*If* $\mathcal{C}$ *is a 1-generator GQC code of length* $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ *over* R *generated by* $X_1 = (\xi_1(g_1 + uk_1 + vp_1 + uvq_1 + wr_1 + uws_1 + vwy_1 + uvwz_1), \dots, \xi_\ell(g_\ell + uk_\ell + vp_\ell + uvq_\ell + wr_\ell + uws_\ell + vwy_\ell + uvwz_\ell))$ *where* $\xi_i, g_i, k_i, q_i, r_i, s_i, y_i, z_i \in F_2[x]$, g_i *divides* $x^{\lambda_i} - 1$ *for* $1 \le i \le \ell$. *Also* $\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i$ *divides* $x^{\lambda_i} - 1$, $p_i = \frac{x^{\lambda_i}-1}{\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i}$ *such that* $gcd(\xi_i, p_i) = 1$ *for each* $1 \le i \le \ell$ *if* $p = lcm(p_1, p_2, \dots p_\ell)$ *with deg(p)=m−*t_1, *then* $\mathcal{C}$ *is free module and minimal spanning set is* $Z_1 = \{X_1, xX_1, \dots, x^{m-t_1-1}X_1\}$ *and* $|c| = 2^{8(m-t_1-1)}$. *Also, d(c)* $\ge \sum \eta_i$ *where* δ *= max*$X \subseteq (1,2, \dots, l)$ *and lcm* p_i *and* η_i *is minimum distance of* $\Pi_i(C)$

Further if $p_1 = p_2 = \cdots = p_\ell$ then $d(C) \ge \Sigma\eta_i$

Proof. Following with the same arguments given in theorem 3.3, $\mathcal{C}$ is free module and minimal spanning set is $Z_1 = \{X_1, xX_1, \dots, x^{m-t_1-1}X_1\}$ and $|c| = 2^{8(m-t_1-1)}$. If c(x)=$(c_1(x), c_2(x), \dots, c_l(x)) \in C$. Then c(x)= ξ(x)X for some $\xi(x) \in R[x]$ and if c(x) = 0 for $1 \le i \le \ell$, implies that $x^{\lambda_i} - 1$ divides $\xi(x)(\xi_i g_i + uk_i + vp_i + uvq_i + wr_i + uws_i + vwy_i + uvwz_i)$ and hence p_i divides $\xi(x)$ for $1 \le i \le \ell$. Therefore c(x) $\ne$ 0

if and only if p divides $\xi(x)$ for $1 \leq i \leq \ell$. In other words when p divides $\xi(x)$, C has maximum number of non- zero co-ordinate positions and their distances is greater than minimum distance of corresponding projections. So d(c) $\geq \sum \eta_i$ for δ = max X $\subseteq (1,2,\ldots,\ell)$ and lcm p_i . Further, if $\delta = \phi$ for any $1 \leq i \leq \ell$ then $d(C) \geq \Sigma\eta_i$.

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