

r-set and r*-set via δ -open set

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Abstract

The notion of δ -open sets in a topological space was studied by Velicko. Following this Ekici et. al. studied the notions of α -open and e^* -open sets by mixing the operators closure, interior, δ -interior and δ -closure. In this paper some new sets are defined and studied by using the above operators. Moreover some of the existing concepts in topological spaces have been characterized using the newly defined sets.

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1.1. Introduction

Stone introduced the concept of regular open sets in the year 1937. Velicko introduced the notion of δ -open sets. Following this, the above sets were extensively investigated by several topologists. In this paper the interior and closure operators in a topological space (X, τ) and its semi-regularization (X, τ^δ) are used to define an r -set and an r^* -set in (X, τ) . Some of the nearly open sets and nearly closed sets have been characterized by using r -sets and r^* -sets. A brief survey of basic concepts and results that are needed here is given in this Section-1. The Section-2 deals with the newly defined sets namely r -set and r^* -set and their characterizations.

1. Preliminaries

The interior and closure operators in topological spaces play a vital role in the generalization of open sets and closed sets in topological spaces. The authors[2] discussed the application of these operators to some nearly open and nearly closed sets.

Definition 1.1. A subset A of a space X is called

- (i) regular open[11] if $A = \text{IntCl}A$ and regular closed if $A = \text{ClInt}A$,
- (ii) α -open[7] if $A \subseteq \text{IntClInt}A$ and α -closed if $\text{ClIntCl}A \subseteq A$
- (iii) semiopen[5] if $A \subseteq \text{ClInt}A$ and semiclosed if $\text{IntCl}A \subseteq A$,
- (iv) preopen[6] if $A \subseteq \text{IntCl}A$ and preclosed if $\text{ClInt}A \subseteq A$,
- (v) semi-preopen [3] or β -open [1] if $A \subseteq \text{ClIntCl}A$ and semi-preclosed or β -closed if $\text{IntClInt}A \subseteq A$,

The operators $sClA$, $pClA$, αClA , βClA , $sIntA$, $pIntA$, $\alpha IntA$, and $\beta IntA$ may defined in an usual way. Andrijevic [3] established the next lemma.

Lemma 1.2:

- (i) $\alpha ClA = A \cup \text{ClIntCl}A$ and $\alpha IntA = A \cap \text{IntClInt}A$.
- (ii) $sClA = A \cup \text{IntCl}A$ and $sIntA = A \cap \text{ClInt}A$.
- (iii) $pClA = A \cup \text{ClInt}A$ and $pIntA = A \cap \text{IntCl}A$.
- (iv) $\beta ClA = A \cup \text{IntClInt}A$ and $\beta IntA = A \cap \text{ClIntCl}A$.

The notion of δ -closure was introduced and studied by Velicko [13]. A point x is in the δ -closure of A if every regular open nbd of x intersects A . $Cl_\delta A$ denotes the δ -closure of A .

Definition 1.3: A subset A of a space X is δ -closed if $A = Cl_\delta A$. The complement of a δ -closed set is δ -open. The collection of all δ -open sets is a topology denoted by τ^δ . This τ^δ is called the semi - regularization of τ . Let $Int_\delta A$ be the interior of A in (X, τ^δ) , called the δ -interior of A . The next lemma is due to Velicko[13].

Lemma 1.4:

- (i) For any open set A , $Cl_\delta A = ClA$
- (ii) For any closed set B , $Int_\delta B = IntB$.

Definition 1.5: A subset A of a space (X, τ) is called

- (i) δ -open [4] if $A \subseteq IntClInt_\delta A$
- (ii) δ^* -open [4] if $A \subseteq ClIntCl_\delta A$
- (iii) δ -semiopen[9] if $A \subseteq ClInt_\delta A$ and δ -semiclosed if $IntCl_\delta A \subseteq A$,
- (iv) δ -preopen[10] if $A \subseteq IntCl_\delta A$ and δ -preclosed if $ClInt_\delta A \subseteq A$,

Definition 1.6: A space X is called locally indiscrete if every open set is closed .

Noiri [8] and Thangavelu and Rao [12] established the following lemmas respectively.

Lemma 1.7 : If A or B is semiopen then $IntCl(A \cap B) = IntClA \cap IntClB$.

Lemma 1.8 : If A or B is semiclosed then $ClInt(A \cup B) = ClIntA \cup ClIntB$.

Throughout this paper (X, τ) is a topological space.

2. δ -set and δ^* -set

Throughout this section A, B are subsets of a topological space (X, τ) .

Proposition 2.1: The following always hold.

- (i) $IntClA = Int_\delta ClA \subseteq Int_\delta Cl_\delta A = IntCl_\delta A$.
- (ii) $Cl_\delta Int_\delta A = ClInt_\delta A \subseteq ClIntA = Cl_\delta IntA$.

Proof: $Int_\delta A \subseteq IntA \Rightarrow Int_\delta ClA \subseteq IntClA$ by replacing A by ClA .

$ClA \subseteq Cl_\delta A \Rightarrow IntClA \subseteq IntCl_\delta A$. Then it follows that

$$Int_\delta ClA \subseteq IntClA \subseteq IntCl_\delta A \quad (\text{Exp.2.1})$$

Taking interior and δ -interior on both side of $ClA \subseteq Cl_\delta A$ we get $IntClA \subseteq IntCl_\delta A$ and

$Int_\delta ClA \subseteq Int_\delta Cl_\delta A \subseteq IntCl_\delta A$ which implies

$$Int_\delta ClA \subseteq Int_\delta Cl_\delta A \subseteq IntCl_\delta A. \quad (\text{Exp.2.2})$$

Using Lemma 1.8 we have $Int_\delta ClA = IntClA$ and $Int_\delta Cl_\delta A = IntCl_\delta A$. Then from (Exp. 2.1) and (Exp.2.2), it follows that $IntClA = Int_\delta ClA \subseteq Int_\delta Cl_\delta A = IntCl_\delta A$ that proves (i). The assertion (ii) can be analogously proved.

Proposition 2.2:

- (i) If A is open then $Int_\delta ClA = Int_\delta Cl_\delta A = IntClA = IntCl_\delta A$.
- (ii) If A is closed then $ClInt_\delta A = ClIntA = Cl_\delta IntA = Cl_\delta IntA$

Proof: Suppose A is open. Then using Lemma 1.4(i), $Cl_\delta A = ClA$ that implies $Int_\delta Cl_\delta A = Int_\delta ClA$. By using Proposition 2.1(i), we have $IntCl_\delta A = Int_\delta Cl_\delta A = Int_\delta ClA = IntClA$. This proves (i) and the proof for (ii) is analogous.

Remark 2.3: The conclusion in Proposition 2.2(i) holds even if A is not open and that in Proposition 2.2(ii) holds even if A is not closed as given in the following example.

Example 2.4: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Then

$\tau' = \{\emptyset, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{a, d\}, \{d\}, X\}$. It can be verified that

$\delta O = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$, $\delta C = \{\emptyset, \{b, c, d\}, \{a, d\}, \{d\}, X\}$.

$Int_{\delta} Cl\{b, d\} = Int_{\delta} Cl_{\delta}\{b, d\} = Int\ Cl\{b, d\} = Int\ Cl_{\delta}\{b, d\} = \{b, c\}$ even though $\{b, d\}$ is not open. Also $ClInt_{\delta}\{a\} = ClInt\{a\} = Cl_{\delta}Int_{\delta}\{a\} = Cl_{\delta}Int\{a\} = \{a, d\}$ even though $\{a\}$ is not closed.

The above remark and example motivate us to have following definition.

Definition 2.5: A subset A of a space (X, τ) is an r -set if $IntCl_{\delta}A = IntClA$ and an r^* -set if $ClInt_{\delta}A = ClIntA$.

Proposition 2.6: A subset A of a space (X, τ) is an r -set if and only if $X \setminus A$ is an r^* -set.

Proof: A is an r -set $\Leftrightarrow IntCl_{\delta}A = IntClA \Leftrightarrow X \setminus IntCl_{\delta}A = X \setminus IntClA \Leftrightarrow ClInt_{\delta}(X \setminus A) = ClInt(X \setminus A) \Leftrightarrow X \setminus A$ is an r^* -set.

Proposition 2.7:

- (i) If A is open or regular open or δ -open then it is an r -set.
- (ii) If A is closed or regular closed or δ -closed then it is an r^* -set.

Proof: Follows from Proposition 2.2.

Proposition 2.8:

- (i) A is an r -set $\Leftrightarrow Int_{\delta}ClA = Int_{\delta}Cl_{\delta}A = IntClA = IntCl_{\delta}A$.
- (ii) A is an r^* -set $\Leftrightarrow ClInt_{\delta}A = ClIntA = Cl_{\delta}Int_{\delta}A = Cl_{\delta}IntA$.

Proof: Follows from Definition 2.5 and Proposition 2.2.

Lemma 2.9:

- (i) $IntCl_{\delta}A = IntCl_{\delta}IntCl_{\delta}A$.
- (ii) $Int_{\delta}ClA = Int_{\delta}ClInt_{\delta}ClA$.
- (iii) $ClInt_{\delta}A = ClInt_{\delta}ClInt_{\delta}A$.
- (iv) $Cl_{\delta}IntA = Cl_{\delta}IntCl_{\delta}IntA$.

Proof: Let A be a sub set of a topological space X . Then

$Cl_{\delta}A = Cl_{\delta}A \Rightarrow IntCl_{\delta}A \subseteq Cl_{\delta}A \Rightarrow Cl_{\delta}IntCl_{\delta}A \subseteq Cl_{\delta}A$. This implies $IntCl_{\delta}IntCl_{\delta}A \subseteq IntCl_{\delta}A$. (Exp.2.3)

$IntCl_{\delta}A \subseteq Cl_{\delta}IntCl_{\delta}A$ that implies $IntCl_{\delta}A \subseteq IntCl_{\delta}IntCl_{\delta}A$. (Exp.2.4)

From Exp.2.3 and Exp.2.4, it follows that $IntCl_{\delta}A = IntCl_{\delta}IntCl_{\delta}A$ that proves (i). Now

$$\begin{aligned} Int_{\delta}A = Int_{\delta}A &\Rightarrow Int_{\delta}A \subseteq ClInt_{\delta}A \Rightarrow Int_{\delta}A \subseteq Int_{\delta}ClInt_{\delta}A \\ &\Rightarrow ClInt_{\delta}A \subseteq ClInt_{\delta}ClInt_{\delta}A \end{aligned} \quad \text{(Exp.2.5)}$$

$$\begin{aligned} ClInt_{\delta}A = ClInt_{\delta}A &\Rightarrow Int_{\delta}ClInt_{\delta}A \subseteq ClInt_{\delta}A \\ &\Rightarrow ClInt_{\delta}ClInt_{\delta}A \subseteq ClInt_{\delta}A \end{aligned} \quad \text{(Exp.2.6)}$$

From Exp.2.5 and Exp.2.6, we see that $ClInt_{\delta}A = ClInt_{\delta}ClInt_{\delta}A$ that proves (iii). The proof for (ii) and (iv) is analog.

Prproposition 2.10:

- (i) $IntCl_\delta A$ is both regular open and an r-set.
- (ii) $ClInt_\delta A$ is both regular closed and an r^* -set.

Proof: Since the interior of a closed set is regular open and since $Cl_\delta A$ is always closed, it follows that $IntCl_\delta A$ is regular open. Now using Proposition 2.9 we have

$IntCl_\delta IntCl_\delta A = IntCl_\delta A$. Since $IntCl_\delta A$ is regular open we have

$IntCl_\delta(IntCl_\delta A) = IntCl_\delta A = IntCl(IntCl_\delta A)$ that implies $IntCl_\delta A$ is an r-set. This proves (i) and the rest can be proved by taking complement.

Let B be an r-set in the next five theorems.

Theorem 2.11: The following are equivalent.

- (i) B is preopen.
- (ii) B is preopen in (X, τ^δ) .
- (iii) B is δ -preopen.
- (iv) $B \subseteq Int_\delta ClB$.

Proof: Suppose B is an r-set. Then using Proposition 2.8, we have

$Int_\delta ClB = Int_\delta Cl_\delta B = IntClB = IntCl_\delta B$. Therefore $B \subseteq IntClB \Leftrightarrow B \subseteq Int_\delta Cl_\delta B \Leftrightarrow B \subseteq IntCl_\delta B \Leftrightarrow B \subseteq Int_\delta ClB$.

This proves the theorem.

Corollary 2.12: The following are equivalent.

- (i) B is semiclosed.
- (ii) B is semiclosed in (X, τ^δ) .
- (iii) B is δ -semiclosed.
- (iv) $Int_\delta ClB \subseteq B$.

Theorem 2.13: The following are equivalent.

- (i) B is α -closed.
- (ii) $ClInt_\delta Cl_\delta B \subseteq B$
- (iii) B is a-closed.
- (iv) B is $ClInt_\delta ClB \subseteq B$

Proof: Suppose B is an r-set. Then using Proposition 2.8, we have

$Int_\delta ClB = Int_\delta Cl_\delta B = IntClB = IntCl_\delta B$ that implies $ClInt_\delta ClB = ClInt_\delta Cl_\delta B = ClIntClB = ClIntCl_\delta B$. Therefore $ClIntClB \subseteq B \Leftrightarrow ClInt_\delta Cl_\delta B \subseteq B \Leftrightarrow ClIntCl_\delta B \subseteq B \Leftrightarrow ClInt_\delta ClB \subseteq B$. This proves the theorem.

Corollary 2.14: The following are equivalent.

- (i) B is β -open
- (ii) $B \subseteq ClInt_\delta Cl_\delta B$
- (iii) B is e^* -open.
- (iv) $B \subseteq ClInt_\delta ClB$.

Theorem 2.15:

- (i) $sClB = B \cup Int_\delta ClB = B \cup Int_\delta Cl_\delta B = B \cup IntCl_\delta B$.
- (ii) $pIntB = B \cap Int_\delta ClB = B \cap Int_\delta Cl_\delta B = B \cap IntCl_\delta B$.
- (iii) $\alpha IntB = B \cap Int_\delta Cl Int B = B \cap Int_\delta Cl_\delta Int B = B \cap IntCl_\delta Int B$.
- (iv) $\alpha ClB = B \cup Cl Int_\delta ClB = B \cup Cl Int_\delta Cl_\delta B = B \cup ClIntCl_\delta B$.
- (v) $\beta IntB = B \cap Cl Int_\delta ClB = B \cap Cl Int_\delta Cl_\delta B = B \cap ClIntCl_\delta B$.
- (vi) $\beta ClB = B \cup Int_\delta ClIntB = B \cup Int_\delta Cl_\delta IntB = B \cup IntCl_\delta IntB$.

Proof: Let B be an r-set. Then we have

$$IntClB = Int_{\delta}ClB = Int_{\delta}Cl_{\delta}B = IntCl_{\delta}B \quad (\text{Exp.2.7})$$

Replacing B by $IntB$ in Exp.2.7 we get

$$IntClIntB = Int_{\delta}ClIntB = Int_{\delta}Cl_{\delta}IntB = IntCl_{\delta}IntB \quad (\text{Exp.2.8})$$

Taking closure on Exp.2.7 we get

$$ClIntClB = ClInt_{\delta}ClB = ClInt_{\delta}Cl_{\delta}B = ClIntCl_{\delta}B \quad (\text{Exp.2.9})$$

Then the proof follows from Exp.2.7, Exp.2.8 and Exp.2.9.

Let B be an r^* -set in the next five theorems.

Theorem 2.16:

- (i) $sIntB = B \cap ClInt_{\delta}B = B \cap Cl_{\delta}Int_{\delta}B = B \cap Cl_{\delta}IntB.$
- (ii) $pClB = B \cup ClInt_{\delta}B = B \cup Cl_{\delta}Int_{\delta}B = B \cup Cl_{\delta}IntB.$
- (iii) $\alpha IntB = B \cap IntClInt_{\delta}B = B \cap IntCl_{\delta}Int_{\delta}B = B \cap IntCl_{\delta}IntB.$
- (iv) $\alpha ClB = B \cup ClInt_{\delta}ClB = B \cup Cl_{\delta}Int_{\delta}ClB = B \cup Cl_{\delta}IntClB.$
- (v) $\beta IntB = B \cap ClInt_{\delta}ClB = B \cap Cl_{\delta}Int_{\delta}ClB = B \cap Cl_{\delta}IntClB.$
- (vi) $\beta ClB = B \cup IntClInt_{\delta}B = B \cup IntCl_{\delta}Int_{\delta}B = B \cup IntCl_{\delta}IntB.$

Proof: Let B be an r^* -set. Then we have

$$ClInt_{\delta}B = ClIntB = Cl_{\delta}Int_{\delta}B = Cl_{\delta}IntB. \quad (\text{Exp.2.10})$$

Replacing B by ClB in Exp.2.10 we get

$$ClInt_{\delta}ClB = ClIntClB = Cl_{\delta}Int_{\delta}ClB = Cl_{\delta}IntClB. \quad (\text{Exp.2.11})$$

Taking interior on Exp.2.10 we get

$$IntClInt_{\delta}B = IntClIntB = IntCl_{\delta}Int_{\delta}B = IntCl_{\delta}IntB \quad (\text{Exp.2.12})$$

Then the proof follows from Exp. 2.10, Exp.2.11 and Exp.2.12.

Theorem 2.17: The following are equivalent.

- (i) B is semiopen.
- (ii) B is semiopen in (X, τ^{δ}) .
- (iii) B is δ -semiopen.
- (iv) $B \subseteq Cl_{\delta}IntB.$

Proof: Suppose B is an r^* -set. Then using Proposition 2.8, we have

$$ClInt_{\delta}B = ClIntB = Cl_{\delta}Int_{\delta}B = Cl_{\delta}IntB. \text{ Therefore } B \subseteq ClIntB \Leftrightarrow B \subseteq Cl_{\delta}Int_{\delta}B \Leftrightarrow B \subseteq ClInt_{\delta}B \Leftrightarrow B \subseteq Cl_{\delta}IntB.$$

This proves the theorem.

Corollary 2.18: The following are equivalent.

- (v) B is preclosed .
- (vi) B is preclosed in (X, τ^{δ}) .
- (vii) B is δ -preclosed.
- (viii) $Int_{\delta}ClB \subseteq B.$

Theorem 2.19: The following are equivalent.

- (i) B is β -closed .
- (ii) e^* -closed
- (iii) $IntCl_{\delta}Int_{\delta}B \subseteq B$
- (iv) $IntCl_{\delta}IntB \subseteq B$

Proof: Suppose B is an r^* -set. Then using Proposition 2.8, we have

$$ClInt_{\delta}B = ClIntB = Cl_{\delta}Int_{\delta}B = Cl_{\delta}IntB \text{ so that } IntClInt_{\delta}B = IntClIntB = IntCl_{\delta}Int_{\delta}B$$

$= IntCl_{\delta}IntB$. Therefore $IntClIntB \subseteq B \Leftrightarrow IntClInt_{\delta}B \subseteq B \Leftrightarrow IntCl_{\delta}Int_{\delta}B \subseteq B \Leftrightarrow IntCl_{\delta}IntB \subseteq B$. This proves the theorem.

Corollary 2.20: The following are equivalent.

- (i) B is α -open
- (ii) B is a -open
- (iii) $B \subseteq IntCl_{\delta}Int_{\delta}B$
- (iv) $B \subseteq IntCl_{\delta}IntB$

Proposition 2.21: Let (X, τ) be locally indiscrete.

- (i) B is an r -set $\Leftrightarrow ClB = Cl_{\delta}B$.
- (ii) B is an r^* -set $\Leftrightarrow IntB = Int_{\delta}B$

Proof : Suppose X is locally indiscrete. Then $IntClB = ClB$ and $IntCl_{\delta}B = Cl_{\delta}B$. Therefore B is an r -set $\Leftrightarrow IntClB = IntCl_{\delta}B \Leftrightarrow ClB = Cl_{\delta}B$.

Also $ClIntB = IntB$ and $ClInt_{\delta}B = Int_{\delta}B$ that implies B is an r^* -set $\Leftrightarrow ClIntB = ClInt_{\delta}B \Leftrightarrow IntB = Int_{\delta}B$.

Proposition 2.22: Let A and B be r -sets in (X, τ) . Then $A \cap B$ is an r -set if one of them is semiopen .

Proof: Let A and B be any two r -sets in (X, τ) and A be semiopen. Therefore

$$IntClA = IntCl_{\delta}A \text{ and } IntClB = IntCl_{\delta}B \text{ that implies}$$

$$IntCl(A \cap B) \subseteq IntCl_{\delta}(A \cap B) \subseteq IntCl_{\delta}A \cap IntCl_{\delta}B = IntClA \cap IntClB . \text{ Now using Lemma 1.7 we have}$$

$$IntClA \cap IntClB = IntCl(A \cap B) = IntCl_{\delta}(A \cap B) \text{ that implies } A \cap B \text{ is an } r\text{-set.}$$

Proposition 2.23: Let A and B are r^* -sets in (X, τ) . Then $A \cup B$ is an r^* -set if one of them is semiclosed.

Proof: Let A and B be any two r^* -sets in (X, τ) and A be semiclosed. Therefore

$$ClIntA = ClInt_{\delta}A \text{ and } ClIntB = ClInt_{\delta}B. \text{ Since } A \text{ is semiclosed we have}$$

$$ClInt_{\delta}(A \cup B) \subseteq ClInt(A \cup B) = ClIntA \cup ClIntB = ClInt_{\delta}A \cup ClInt_{\delta}B \subseteq ClInt_{\delta}(A \cup B).$$

Therefore $ClInt_{\delta}(A \cup B) = ClInt(A \cup B)$ that implies $A \cup B$ is an r^* -set.

Conclusion

The two level operators in topology namely $IntCl_{\delta}A$ and $ClInt_{\delta}A$ are used to define new sets in topology namely r -set and r^* -set. Some existing sets in topology namely regular open, semiopen, preopen, α -open, β -open, δ -semiopen, δ -preopen, a -open, e^* -open sets and their corresponding closed sets are characterized using the newly defined sets. Moreover it has been established that the intersection of two r -sets is again an r -set if atleast one of them is semiopen and the intersection of two r^* -sets is an r^* -set if atleast one of them is semiclosed.

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