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A New Class of Univalent Harmonic Meromorphic Functions Defined by Multiplier Transformation

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Abstract:

The seminal work of Clunie and Sheil-Small [4] on harmonic mappings gave rise to studies on subclasses of complex valued harmonic univalent functions. In this paper we introduced new family of analytical function $f(z)=h(z)+\overline{g(z)}$, which is harmonic meromorphic functions of complex order in the open disk $\overline{U}=\{z:|z|>1\}$ defined by using modified Salagean operator. It is shown that the functions in this class are sense preserving and univalent outside the unit disk. Sufficient conditions are obtained for functions in this class which are also shown to be necessary when the co-analytic part $g(z)$ has negative coefficients. We also obtain properties such as distortion bounds, extreme points, convolution and convex combination for this class.

Keywords: Harmonic Functions, Meromorphic Functions, Univalent Functions, Starlike Functions, Modified Multiplier Transformation.

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I. Introduction

Harmonic univalent mappings are known to play an important role in the study of minimal surfaces and have found applications in different fields such as Engineering,

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Operation research and applied mathematics [2]. Harmonic mappings in the domain $D \subseteq C$ are univalent complex valued harmonic functions $f = u + iv$ where both u and v real harmonic in D. Harmonic univalent mappings have drawn tremendous attention of complex analysis only after the important work of Clunie and Sheil-Small [4] in 1984. Hengartner and Schober [7] and [8] in 1986 worked towards finding an appropriate form of the Riemann mapping theorem for harmonic mappings. The works of these function theorists and several other researchers (see for example [9],[18],[19]) gave rise to several problems, conjectures and many intriguing questions. Several classes of complex valued harmonic univalent functions have been introduced and investigated following the basic work of Clunie and Sheil-Small [4]. There are several survey articles and books ([2],[5]) on harmonic mappings and related areas as ([13],[17]). Hengartner and Schober [9], among other things, investigated the family M of functions $f(z) = h(z) + \overline{g(z)}$ which are harmonic, Meromorphic, orientation preserving and univalent in $\overline{U} = \{z : |z| > 1\}$ where

$$\mathrm{h}(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad ; \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k} \quad , \ z \in \overline{U} \tag{1}$$

Jahangiri [10] and Jahangiri and Silverman [12] have also investigated harmonic, Meromorphic functions which are starlike in $\overline{U}$.

For this class the function $f(\mathrm{z})$ may be expressed as

$$\mathrm{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Cho and Srivastava [3], introduced the operator $I_\lambda^n : A \to A$ defined as

$$I_\lambda^n \, \mathrm{f(z)} = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n a_k z^k,$$

Where A denote the class of functions of the form (2) which are analytic in the open unit disc U.

For $\lambda = 1$, the operator $I_\lambda^n \equiv I^n$ was studied by Uralegaddi and Somanatha [20] and for $\lambda = 0$ the operator I_λ^n reduce to well-known Salagean operator introduced by Salagean [16].

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Definition 1.1 Let, $f(z)=h(z)+\overline{g(z)}$ be a function, where h and g are of the form (1), the modified multiplier transformation for f defined in [11] as:

$$I_\lambda^n \mathrm{f(z)} = I_\lambda^n \mathrm{h(z)} + (-1)^n \overline{I_\lambda^n \mathrm{g(z)}}, \qquad n \in N_0 = N \cup \{0\} \tag{3}$$

Where,

$$I_\lambda^n h(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right) a_k z^k, \qquad I_\lambda^n g(z) = \sum_{k=1}^{\infty} \left(\frac{k-\lambda}{1+\lambda} \right) \overline{b_k z^k}.$$

When we put $\lambda = 0$, we get modified Salagean operator introduced in [11].

Definition 1.2 In this paper, motivated by study in [14], Here we define a class $S_H(b, \mathrm{n}, \lambda, \mathrm{t}, \alpha, \gamma)$ of harmonic, meromorphic functions $f = h + \overline{g}$ such that:

$$\operatorname{Re}\left\{ 1 + \frac{1}{b}\left[\left(\frac{\left(1+e^{i\alpha}\right) I_\lambda^{n+1} f(z)}{I_\lambda^n f_t(z)} - e^{i\alpha} \right) - 1 \right] \right\} \geq \gamma \tag{4}$$

where $f_t(z) = (1-t)z + tf(z)$, $0 \le \mathrm{t} \le 1$, $0 \le \gamma < 1$, $0 \le \beta < 1$, α real and b is a complex number such that $|\mathrm{b}| \le 1$.

Remark 1.1: The class includes a variety of well-known subclasses for specific values of b, n, λ, t and α.

1. $when\, b = 1, n = 0, \lambda = 0,\ t = 1,\ S_H(1,0,0,1,\alpha,\gamma) = \mathrm{M}_{\overline{H}}(\gamma)$ [15]
2. $when\, b = 1, n = 0, \lambda = 0,\ S_H(1,0,0,\mathrm{t},\alpha,\gamma) = G_H(\alpha,\beta,t)$ [1]
3. $when\, b = 1, \mathrm{n} = 0,\ \lambda = 0,\ t = 1, \alpha = 0,\ S_H(1,0,0,1,0,\gamma) = \sum_H^* \left(\frac{1+\beta}{2} \right)$ [10]
4. $when\ n = 0, \lambda = 0,\ S_H(\mathrm{b},0,0,\mathrm{t},\alpha,\gamma) = S_H(\mathrm{b},\alpha,\gamma,\mathrm{t})$ [6]
5. $when\ \lambda = 0,\ S_H(\mathrm{b},\mathrm{n},0,\mathrm{t},\alpha,\gamma) = S_H(\mathrm{b},\alpha,\gamma,\mathrm{t},\mathrm{n})$ [14]

Also let $S_{\overline{H}}(b, \mathrm{n}, \lambda, \mathrm{t}, \alpha, \gamma)$ be the subclass of $S_H(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ consisting of functions $f = h + \overline{g}$ in which h and g are of the form

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad ; \quad g(z) = -\sum_{k=1}^{\infty} b_k z^{-k} \quad , \ a_k \ge 0,\ b_k \ge 0 \tag{5}$$

We obtain sufficient coefficient conditions for harmonic meromorphic functions $f = h + \overline{g}$ to be in the class $S_H(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$. We also show that this coefficient condition is

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also necessary for $f \in S_{\overline{H}}(b, \mathrm{n}, \lambda, \mathrm{t}, \alpha, \gamma)$ .We also obtain distortion bounds, extreme points, convolution condition and convex combination for functions in $S_{\overline{H}}(b, \mathrm{n}, \lambda, \mathrm{t}, \alpha, \gamma)$.

II. Coefficient Conditions

First, we prove a sufficient condition for harmonic functions in $S_H(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$.

Theorem 2.1: Let $f = h + \overline{g}$ be so that h and g are of the form (1). If

$$\left\{\begin{array}{l} \sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k| \\ +\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k| \end{array}\right\} \leq (1-\gamma)|b| \qquad (6)$$

when $0 \leq \gamma < 1,\ 0 \leq t \leq 1$, α real and b a non-zero complex number such that $|b| \leq 1$, then f is univalent, sense preserving, harmonic mapping in $U = \{z : |z| < 1\}$ and $f \in S_H(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$.

Proof: Consider the function $f = h + \overline{g}$, where h and g are given by (1). In [12] it has been proved that if

$$\sum_{k=1}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 1,\ |b_1| < 1$$

then f is harmonic, orientation preserving and univalent in U. For $0 \leq \gamma < 1$, we note that

$$k \leq \frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}{(1-\gamma)|b|}$$

and

$$k \leq \frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}{(1-\gamma)|b|}.$$

Therefore, f is harmonic, orientation preserving and univalent in $\overline{U}$ due to (6). To show that $f \in S_H(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ we notice according to (4), we must have $\mathrm{Re}\left[\frac{A(z)}{B(z)}\right] > \gamma$ where,

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$$\begin{aligned}A(z) &= b\left[(1-t)z+\left(I_\lambda^n h(z)+(-1)^n I_\lambda^n \overline{g(z)}\right)\right]\\ &\quad+\left(1+e^{i\alpha}\right)\left[\left(I_\lambda^{n+1} h(z)+(-1)^{n+1} I_\lambda^{n+1} \overline{g(z)}\right)\right]\\ &\quad-\left(1+e^{i\alpha}\right)\left[(1-t)z+\left(I_\lambda^n h(z)+(-1)^n I_\lambda^n \overline{g(z)}\right)\right]\\ B(z) &= b\left[(1-t)z+t\left(I_\lambda^n h(z)+(-1)^n I_\lambda^n \overline{g(z)}\right)\right]\end{aligned}$$

Using the fact that, $\operatorname{Re}[\omega]\ge\gamma$ iff $|1-\gamma+w|\ge|1+\gamma-w|$ for $0\le\gamma<1$ it is enough to show that

$$\left|A(z)+(1-\gamma)B(z)\right|-\left|A(z)-(1+\gamma)B(z)\right|\ge 0 \tag{7}$$

Putting equations (1), (3), (4) and (5) in equation (7) we obtain

$$\begin{aligned}&\left|A(z)+(1-\gamma)B(z)\right|-\left|A(z)-(1+\gamma)B(z)\right|\\ &=\left|\begin{array}{l}(2-\gamma)bz-(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t-(2-\gamma)bt\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{-k}\\ -(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n e^{i\alpha}a_k z^{-k}\\ +(-1)^{2n}\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t+(2-\gamma)bt\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n \overline{b_k z^{-k}}\\ +(-1)^{2n}\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n e^{i\alpha}\overline{b_k z^{-k}}\end{array}\right|\\ &\quad-\left|\begin{array}{l}-\gamma bz-(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t+\gamma bt\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{-k}\\ -(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n e^{i\alpha}a_k z^{-k}\\ +(-1)^{2n}\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t-\gamma bt\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n \overline{b_k z^{-k}}\\ +(-1)^{2n}\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n e^{i\alpha}\overline{b_k z^{-k}}\end{array}\right|\end{aligned}$$

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$$\therefore \left|A(z)+(1-\gamma)B(z)\right|-\left|A(z)-(1+\gamma)B(z)\right| \geq \Big\{(2-\gamma)|b||z|$$
$$-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t-(2-\gamma)|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k||z|^{-k}-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left|e^{i\alpha}\right||a_k||z|^{-k}$$
$$-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t+(2-\gamma)bt\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k||z|^{-k}-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left|e^{i\alpha}\right||b_k||z|^{-k}$$
$$-\gamma|b||z|-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t+\gamma|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k||z|^{-k}-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)+t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left|e^{i\alpha}\right||a_k||z|^{-k}$$
$$-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t-\gamma|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k||z|^{-k}-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)-t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left|e^{i\alpha}\right||b_k||z|^{-k}\Big\}$$

$$=(2-\gamma)|b||z|-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+2t-(2-\gamma)|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k||z|^{-k}$$
$$-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-2t+(2-\gamma)|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k||z|^{-k}$$
$$-\gamma bz-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+2t+\gamma|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k||z|^{-k}$$
$$-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-2t-\gamma|b|t\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k||z|^{-k}$$

$$=2\left\{\begin{array}{l}(1-\gamma)|b||z|-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k||z|^{-k}\\ -\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k||z|^{-k}\end{array}\right\}$$

$$\geq 2\left\{(1-\gamma)|b|-\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k|\\ -\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k|\end{array}\right\}\right\}$$

Now by (6), this last expression is never negative and so $f \in S_H(b,\mathrm{n},\lambda,t,\alpha,\gamma)$. We now give an example of a function in the class $S_H(b,\mathrm{n},\lambda,t,\alpha,\gamma)$.

Next, we show that the coefficient condition (8) is also necessary for functions in $S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$.

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Theorem 2.2: Let $f = h + \overline{g}$ be so that h and g are of the form (5). A necessary and sufficient condition for f to be in $f \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ is that

$$\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|a_k| \\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}|b_k|\end{array}\right\} \le (1-\gamma)|b| \tag{8}$$

Proof: In view if above theorem 2.1, we need only show that $f \notin S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ if the coefficient inequality (8) does not hold. We note that if $f \in S_{\overline{H}}(b, \alpha, \gamma, t, n)$ we must have

$$\mathrm{Re}\left\{1+\frac{1}{b}\left[\frac{(1+e^{i\alpha})I_{\lambda}^{n+1}f(z)}{I_{\lambda}^{n}f_t(z)}-e^{i\alpha}-1\right]-\gamma\right\}$$

$$=\mathrm{Re}\left\{1+\frac{1}{b}\left[\frac{(1+e^{i\alpha})\left(I_{\lambda}^{n+1}h(z)+(-1)^{n+1}I_{\lambda}^{n+1}g(z)\right)}{(1\text{-}t)z+t\left(I_{\lambda}^{n}h(z)+(-1)^{n}I_{\lambda}^{n}g(z)\right)}-e^{i\alpha}-1\right]-\gamma\right\}$$

$$=\mathrm{Re}\left\{\frac{\begin{array}{l}b(1-\gamma)z-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)(1+e^{i\alpha})+t\left((1-\gamma)b+(1+e^{i\alpha})\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_k z^{-k} \\ -\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)(1+e^{i\alpha})-t\left((1-\gamma)b+(1+e^{i\alpha})\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_k\overline{z}^{-k}\end{array}}{bz+tb\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_k z^{-k}-tb\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_k\overline{z}^{-k}}\right\}$$

$$=\mathrm{Re}\left\{\frac{\begin{array}{l}|b|^2(1-\gamma)z-\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)(1+e^{i\alpha})+t\left((1-\gamma)b+(1+e^{i\alpha})\right)\right]\overline{b}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_k z^{-k} \\ -\sum_{k=1}^{\infty}\left[\left(\frac{k+\lambda}{1+\lambda}\right)(1+e^{i\alpha})-t\left((1-\gamma)b+(1+e^{i\alpha})\right)\right]\overline{b}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_k\overline{z}^{-k}\end{array}}{|b|^2\left[z+t\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_k z^{-k}-t\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_k\overline{z}^{-k}\right]}\right\}\ge 0$$

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This inequality must hold for all $z \in \overline{U}$ *and for all real* α *and any* b *such that* $|b| = b$.
we have,

$$\operatorname{Re}\left\{\frac{\begin{array}{l}|b|^2(1-\gamma)-\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)b\right)\right]\bar{b}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k r^{-(k+1)}\\ -\sum_{k=1}^{\infty}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)b\right)\right]\bar{b}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k r^{-(k+1)}\end{array}}{|b|^2\left[1+t\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k r^{-(k+1)}-t\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k r^{-(k+1)}\right]}\right\}=\frac{A(r)}{B(r)}\geq 0$$

If the equation (8) does not hold, then $A(r)$ is negative for which the quotient $\frac{A(r)}{B(r)}$ is negative.

This contradicts that $\frac{A(\mathrm{r})}{B(r)} \geq 0$ and so the proof is complete.

III. Distortion Theorem

The distortion bounds for functions in $S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$ are given by Theorem 3.1

Theorem 3.1: If $f \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$ then

$$r-|b|(1-\gamma)r^{-1} \leq |f(z)| \leq r+|b|(1-\gamma)r^{-1}, \quad |z|=r>1.$$

Proof: We prove the right-hand inequality. The argument for the left-hand inequality is similar and hence it omitted. Let $f \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$. Taking the absolute value of f we obtain

$$|f(z)|=\left|z+\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^{-k}-\sum_{k=1}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n b_k \bar{z}^{-k}\right| \leq r+\sum_{k=1}^{\infty}\left(|a_k|+|b_k|\right)\left(\frac{k+\lambda}{1+\lambda}\right)^n r^{-k}$$

$$\leq r+\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n|a_k|\\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n|b_k|\end{array}\right\}r^{-1}$$

$$\leq r+|b|(1-\gamma)r^{-1}.$$

IV. Extreme Points

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We use the coefficient bounds obtained in section 2 to determine the extreme points for functions in $S_{\overline{H}}\left(b,\mathrm{n},\lambda,t,\alpha,\gamma\right)$.

Theorem 4.1: If $f \in S_{\overline{H}}\left(b,\mathrm{n},\lambda,t,\alpha,\gamma\right)$ if and only if f can be expressed as

$$f\left(z\right)=\sum_{k=0}^{\infty}\left(x_k h_k + y_k g_k\right) \; where\, z \in \overline{U},$$

$$\mathrm{h}_0\left(z\right)=z, \qquad g_0\left(z\right)=z,$$

$$h_k\left(z\right)=z+\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n} z^{-k}, \qquad \left(k=1,2,3,..........\right)$$

$$g_k\left(z\right)=z-\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n} \left(\overline{z}\right)^{-k}, \qquad \left(k=1,2,3,..........\right)$$

$$and \quad \sum_{k=0}^{\infty}\left(x_k+y_k\right)=1, \qquad x_k \geq 0,\; y_k \geq 0.$$

Proof: Note that for f we may write,

$$f\left(z\right)=\sum_{k=0}^{\infty}\left(x_k h_k + y_k g_k\right)$$

$$=x_0 h_0 + y_0 g_0 + \sum_{k=1}^{\infty}\left[\begin{array}{l} x_k\left(z+\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n} z^{-k}\right) \\ +y_k\left(z-\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n} |\overline{z}|^{-k}\right)\end{array}\right]$$

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$$= \sum_{n=0}^{\infty}\left(x_k + y_k\right) z + \sum_{n=1}^{\infty}\left[\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}} x_k z^{-k}\right]$$

$$-\sum_{k=1}^{\infty}\left[\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}} y_k \left|\bar{z}\right|^{-k}\right]$$

Now by Theorem 2.2

$$\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left[\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}\right] \\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left[\frac{|b|\left(1-\gamma\right)}{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}\right]\end{array}\right\}$$

$$= |b|\left(1-\gamma\right)\sum_{k=0}^{\infty}\left(x_k + y_k\right) \le |b|\left(1-\gamma\right).$$

Conversely, suppose that $f \in S_{\overline{H}}\left(b, \mathrm{n}, \lambda, t, \alpha, \gamma\right)$ then

$$\sum_{k=1}^{\infty}\left[\begin{array}{l}\frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}{|b|\left(1-\gamma\right)} a_k - \\ \frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-\left(1-\gamma\right)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}}{|b|\left(1-\gamma\right)} b_k\end{array}\right] \le 1.$$

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$$setting,$$

$$x_k = \frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n}{|b|(1-\gamma)} a_k,$$

$$y_k = \frac{\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)|b|\right)\right]\left(\frac{k+\lambda}{1+\lambda}\right)^n}{|b|(1-\gamma)} b_k, \quad 0 \le x_0 \le 1.$$

$$y_0 = 1 - x_0 - \sum_{k=1}^{\infty}(x_k + y_k)$$

We obtain $f(z) = \sum_{k=0}^{\infty}(x_k h_k + y_k g_k)$ as required.

V. Convolution and Convex Combination

In this section we show that the class $S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ is invariant under convolution and convex combinations of its numbers.

For harmonic functions

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} - \sum_{n=k}^{\infty} b_k \left(\overline{z}\right)^{-k}, \qquad F(z) = z + \sum_{k=1}^{\infty} A_k z^{-k} - \sum_{k=1}^{\infty} B_k \left(\overline{z}\right)^{-k}$$

We define the convolution of f and F as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=1}^{\infty} a_k A_k z^{-k} - \sum_{k=1}^{\infty} b_k B_k \left(\overline{z}\right)^{-k}$$

Theorem 5.1: *For* $0 \le \beta \le \gamma \le 1$. *Let* $f \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ *and* $F \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \beta, \gamma)$. *Then* $f * F \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma) \subset S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \beta, \gamma)$.

Proof: Suppose f and F are so that $f * F$ is given by the above convolution.

Since $f \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \alpha, \gamma)$ *and* $F \in S_{\overline{H}}(b, \mathrm{n}, \lambda, t, \beta, \gamma)$, the coefficients of f and F must satisfy conditions given by Theorem 2.2. So, for the coefficients of $f * F$ we can write.

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$$\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} a_k A_k \\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} b_k B_k\end{array}\right\}$$

$$\leq \sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} a_k \\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} b_k\end{array}\right\}$$

The right hand side of the above inequality is bounded by $\mathrm{b}(1-\gamma)$ because $f \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$. Thus $\mathrm{f} * \mathrm{F} \in f \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma) \subset S_{\overline{H}}(b,\mathrm{n},\lambda,t,\beta,\gamma)$.

Finaly we examine the convex combinations of $S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$.

Theorem 5.2: The family $S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$ is closed under convex combination.

Proof: Suppose

$$f_i(z) = z + \sum_{k=1}^{\infty} a_{i_k} z^{-k} - \sum_{n=k}^{\infty} b_{i_k} \left(\bar{z}\right)^{-k} \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$$

where, $a_{i_k} \geq 0,\ b_{i_k} \geq 0$ and i=1,2,3,.....Then by Theorem 2.2,

$$\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} a_{i_k} \\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n} b_{i_k}\end{array}\right\} \leq |b|(1-\gamma)$$

$$\text{For} \sum_{i=1}^{\infty} t_i = 1 \text{ and } 0 \leq t_i \leq 1$$

The convex combination of f_i, may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i_k}\right) z^{-k} - \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{i_k}\right)\left(\bar{z}\right)^{-k}$$

Thus,

$$\sum_{i=1}^{\infty} t_i f_i(z) \in S_{\overline{H}}(b,\mathrm{n},\lambda,t,\alpha,\gamma)$$

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Since

$$\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left(\sum_{i=1}^{\infty}t_i a_{i_k}\right)\\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}\left(\sum_{i=1}^{\infty}t_i b_{i_k}\right)\end{array}\right\}$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=1}^{\infty}\begin{array}{l}\left[2\left(\frac{k+\lambda}{1+\lambda}\right)+t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}a_{i_k}\\ +\left[2\left(\frac{k+\lambda}{1+\lambda}\right)-t\left(2-(1-\gamma)\right)|b|\right]\left(\frac{k+\lambda}{1+\lambda}\right)^{n}b_{i_k}\end{array}\right\}$$

$$\leq\sum_{i=1}^{\infty}t_i|b|(1-\gamma)=|b|(1-\gamma).$$

VI. Conclusion

In this paper an attempt has been made to introduce and investigate some properties for a new subclass of harmonic meromorphic functions of complex order. Based on this work, further useful study on different subclasses of harmonic univalent functions can be established by using different operators.

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