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Facts On The Diophantine Equation $11^X + 2^Y = Z^2$

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Abstract: The paper reveals that (0,3,3) is a unique non-negative integer solution for the Diophantine equation $11^X + 2^Y = Z^2$, where x, y and z are non-negative integers.

Keywords: Catalan Conjectures, Exponential ,Diophantine equations, integer solution

I. Introduction

In 2007, D.Acu (1) evience that (3,0,3) and (2,1,3) are only two solutions in non-negative integers of the Diophantine equation $2^X + 5^Y = Z^2$, In 2013, Rabago (2) proved that $8^X + 19^Y = Z^2$ has the only solutions (x, y, z) = (1,1,5), (2,1,9) and (3,1,23). In 2017, G.J. Komahan (3) substantiate (0,3,3) is a unique non-negative integer solution for the Diophantine equation $27^X + 2^Y = Z^2$. This paper communicates that the Diophantine equation $11^X + 2^Y = Z^2$ has non-negative integer solution.

II. Preliminaries

In the year 1844, Catalan (4) Conjectures that the Diophantine equation $a^X - b^Y = 1$ has a unique integer solution with minimum $\{a, b. x. y\} > 1$. The solution of (a, b, x, y) is (3,2,2,3).Mihailescu (5) proved the conjecture in 2004.

2.1 Preposition [5] For (a,b,x,y),(3,2,2,3) is a unique solution of the Diophantine Equation $a^X - b^Y = 1$, where a, b, x and y are integers with minimum $\{a, b. x. y\} > 1$

2.1 Lemma: (3,3) is unique solution of (y,z) for the Diophantine Equation $1 + 2^y = z^2$, where y and z are non-negative integers

2.2 Lemma: The Diophantine equation $11^x + 1 = z^2$ has no non-negative integer solution where y and z are non-negative integers.

Proof: Claim: x and z are non-negative integers. Let us assume that x and z are non-negative integers such that $11^x + 1 = z^2$

Case (a): x=0, then $11^0 + 1 = z^2$, $z^2 = 2$, which is impossible.

Case (b): $x \geq 1$, then $z^2 = 11^x + 1$, x=1, $z^2 = 2$, $z > 3$ .By Preposition 2.1, we have x=1, then $z^2 = 12$,which is contradiction. Therefore the equation $11^x + 1 = z^2$ has no non-negative integer solution.

III . Results

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3.1 Theorem: $11^x + 2^y = z^2$, where x, y and z are non negative integers has the unique solution (0,3,3).

Proof: Claim: (0,3,3) is the unique solution of $11^x + 2^y = z^2$ .Let x, y and z be non - negative integers such that $11^x + 2^y = z^2$. By lemma 2.2, we have $y \geq 1$.Thus z is Odd, then there is a non-negative integer s such that z=2s+1. Now

$$11^x + 2^y = 4(s^2 + s) + 1.$$

Then $11^x \equiv 1 mod 4$.Therefore x is even. Then there is a non-negative integer L such that x=2L. Let us discuss x in two cases

case I : By lemma 2.1, we have y=3,z=3.

case II : Let $x \geq 2$.

If $L \geq 1$, now $z^2 - 11^{2L} = 2^y$, then

$(z - 11^L)(z + 11^L) = 2^y$ ----------------------- (1)

$(z - 11^L) = 2^v$ ------------------ (2)

where v is a non-negative integer. substituting (2) in (1) becomes

$2^v(z + 11^L) = 2^y$

$(z + 11^L) = 2^{y-v}$ ---------------- (3)

(3) - (2) gives $2(11^L) = 2^v(2^{y-v} - 1)$ ---------------- (4)

Let v be divided into two subcases

Subcase(i) : v=0, it follows that from (2) $(z - 11^L) = 1$ $\therefore$ z is even which is a $\Rightarrow\Leftarrow$

Subcase(ii) **:** v=1, it follows that from (3) $2^{y-2} - 1 = 11^L$, $2^{y-2} > 12$,y>3. By preposition 2.1,since L=1, then $2^{y-2} = 12$. It is impossible. Therefore (0,3,3) is a unique solution of (x, y, z) for $11^x + 2^y = z^2$

Corollary 3.1: The Diophantine equation $11^x + 2^y = t^4$, has no non-negative integer solution where x, y and z are non-negative integers.

Proof: Claim: The Diophantine equation $11^x + 2^y = t^4$, has no non-negative integer for instance $11^x + 2^y = t^4$, where $z = t^2$. By theorem 3.1, we have (x, y, z) = (0,3,3). $\therefore$ $t^2 = z = 3$, which is a contradiction. Hence $11^x + 2^y = t^4$ has no non-negative integer solution.

Corollary 3.2 : (0,3,3) is a unique solution of (x, y, z) for $11^x + 2^u = z^2$, where x, u and z are non-negative integers.

Proof: Claim: x, y and z are non-negative integers. Let us assume x, y and z are non-negative integers such that $11^x + 2^u = z^2$ with $y = u$. By theorem 3.1, we know that (x,y,z) = (0,3,3). $\therefore$ $y = u = 3$, Therefore (0,3,3) is a unique solution for the equation $11^x + 2^u = z^2$.

Corollary 3.3: (0,1,3) is a unique solution of (x, y, z) for the Diophantine equation $11^x + 8^u = z^2$, where x, u and z are non-negative integers.

Proof: Claim: x, y and z are non-negative integers. Let us assume x, y and z are non-negative integers such that $11^x + 8^u = z^2$ with $y = 3u$. By theorem 3.1, (x, y, z) = (0,3, 3). $\therefore y = 3u = 3$, $u = 1$. Therefore (0,3,3) is a unique solution for the equation $11^x + 8^u = z^2$

Corollary 3.3: The Diophantine equation $11^x + 64^u = z^2$, has non-negative integer solution where x, u and z are non-negative integers.

Proof: Claim: The Diophantine equation $11^x + 64^u = z^2$ has no non-negative integer. Assume x , u and z are non-negative integers such that $11^x + 64^u = z^2$, with **y**$= 64$. Then by theorem 3.1 , $y = 6u = 3$, which is a contradiction. ∴ x , y and z has no non negative integers.

Conclusion: The Diophantine equation $11^x + 2^y = z^2$ has a unique non negative integer solution (0,3,3)

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