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Controllability Of Sobolev Type Semilinear Integro-Differential Evolution Systems In A Separable Banach Space

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Abstract

In this paper, we study the controllability of certain evolution differential systems, sufficient conditions ensuring the controllability of semilinear neutral evolution integro differential systems with nonlocal initial conditions in a separable Banach space. The results are obtained by using Hausdorff measure of noncompactness and a new calculation methods are employed for achieving the required results.

Keywords: *Controllability, semilinear differential system, evolution system, measure of noncompactness.*

1 INTRODUCTION

In science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models. Popular models essentially fall into two categories: the differential models and the integrodifferential models. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. In general functional differential equations or evolution equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory and many other physical phenomena.

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional spaces has been extensively investigated. The problem of controllability of linear systems represented by differential equations in Banach spaces has been extensively studied by several authors [10]. Several papers have appeared on finite dimensional controllability of linear systems [13] and infinite dimensional systems in abstract spaces [9]. Of late the controllability of nonlinear systems in finite-dimensional spaces is studeid by means of fixed point principles [1]. Several authors have extended the concept of controllability to infinite-dimensional spaces by applying semigroup theory [8, 18, 23, 24]. Controllability of nonlinear systems with different types of nonlinearity has been studied by many authors with the help of fixed point principles [2]. Naito [17] discussed the controllability of nonlinear Volterra integrodifferential systems and in [15, 16] he studied the controllability of semilinear systems whereas Yamamoto and Park [22] investigated the same problem for a parabolic equation with a uniformly bounded nonlinear term.

2 PRELIMINARIES

Consider the class of sobolev-type semilinear neutral functional integrodifferential system with nonlocal conditions of the form

$$\frac{d}{dt}[Ex(t) - g(t, x(t))] = A(t)x(t) + Bu(t) + f\left(t, x(t), \int_0^t k\big(t, s, x(s)\big)ds, \int_0^a h\big(t, s, x(s)\big)ds\right), t \in [0, b], \quad (2.1)$$

$$x(t) + q(x) = \phi(t) \quad t \in [-r, 0] \quad (2.2)$$

where the state variable $x(\cdot)$ takes values in a separable Banach space X with norm $\|\cdot\|$ and the control function $u(\cdot)$ is given in $\mathcal{L}^2(I, U)$, a Banach space of admissible control functions with U as a

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Banach space the interval $I = [0, b]$. E and B is a bounded linear operator from U into X and $A(t): D_t \subset X \to X$ generates an evolution system $\{U(t,s)\}_{0 \le s \le t \le b}$ on the separable Banach space X. The functions $g: I \times C \to X$, $f: I \times C \times X \times X \to X$, $k: I \times I \times C \to X$ $h: I \times I \times C \to X$, $g: \mathcal{C}(I, X) \to X$ are given functions. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi: [-r, 0] \to X$ endowed with the norm $\| \phi \| = sup\{|\phi(\theta)|: -r \le \theta \le 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b], x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The norm of the X is denoted by $\|\dot{\|}$.

Throughout this paper, $\{A(t): t \in \mathbb{R}\}$ is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in X and we assume that the linear non-autonomous system

$$u'(t) = A(t)u(t), \quad s \le t \le b,$$
$$u(s) = x \in X, \tag{2.3}$$

has associated evolution family of operators $\{U(t,s): 0 \le s \le t \le b\}$. In the next definition, $\mathcal{L}(X)$ is a space of bounded linear operators from X into X endowed with the uniform convergence topology.

A family of operators $\{U(t,s): 0 \le s \le t \le b\} \subset \mathcal{L}(X)$ is called a evolution family of operators for (2.3) if the following properties hold:

1. $U(t,s)U(s,\tau) = U(t,\tau)$ and $U(t,t)x = x$, for every $s \le \tau \le t$ and all $x \in X$;
2. For each $x \in X$, the function for $(t,s) \to U(t,s)x$ is continuous and $U(t,s) \in \mathcal{L}(X)$ for every $t \ge s$ and
3. For $0 \le s \le t \le b$, the function $t \to U(t,s)$, for $(s,t] \in \mathcal{L}(X)$, is differentiable with $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$.

[19, 20] System $(2.1) - (2.2)$ is said to be *controllable* on the interval J, if for every initial functions $x_0 \in X$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of $(2.1) - (2.2)$ satisfies $x(0) = x_0$ and $x(b) = x_1$.

A solution $x(\cdot) \in \mathcal{C}([0,b], X)$ is said to be a mild solution of $(2.1) - (2.2)$, then the following integral equation is satisfied.

$$x(t) = E^{-1}U(t,0)[E\phi(0) - Eq(x) - g(0,x_0)] + E^{-1}g(t,x_t)$$
$$+ \int_0^t E^{-1}U(t,s)Cu(s)ds + \int_0^t E^{-1}U(t,s)A(s)g(s,x_s)ds$$
$$+ \int_0^t E^{-1}U(t,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau, \int_0^a h(s,\tau,x_\tau)d\tau)ds, \quad t \in I.$$
$$x(t) + q(x) = \phi(t) \quad\quad t \in [-r, 0]$$

We need the following fixed point theorem due to Schaefer [19]

Theorem 1. Let E be a normed linear space. Let $F: E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let $\zeta(F) = \{x \in E: x = \lambda F x \quad for\ some \quad 0 < \lambda < 1\}$. Then either $\zeta(F)$ is unbounded or F has a fixed point.

To study the controllability problem, we assume the following hypotheses:

(H1). $A(t)$ generates a strongly continuous semigroup of a family of evolution operators $U(t,s)$ and there exist constants $M_1 > 0$ such that
$$\| U(t,s) \| \le M_1, \quad\quad for \quad 0 \le s \le t \le b,$$

(H2). There exists a positive constant $0 < b_0 < b$ and, for each $0 < t \le b_0$, there is a compact set $V_t \subset X$ such that $U(t,s)f(s,x_s,\int_o^s k(s,\tau,x_\tau)d\tau, \int_0^a h(s,\tau,x_\tau)d\tau)$, $U(t,s)A(s)g(s,x_s)$, $U(t,s)Bu(s) \in V_t$ and all $0 \le \tau \le s \le b_0$.

(H3). The linear operator $W: L^2(I,U) \to X$ defined by $Wu = \int_0^b U(b,s)Cu(s)ds$ has an inverse operator W^{-1} which takes values in $L^2(I,U)/kerW$ and there exists a positive constant M_2 such that $\| CW^{-1} \| \le M_2$.

(H4). (a) The function $g: I \times X \to X$ is continuous for a.e. $t \in I$ and there exists a positive constant $M_g > 0, L_g > 0$ such that
$$\| g(t,x_t) \| \le M_g \| x_t \|, \text{ and } \| g(0,x_0) \| \le L_g$$

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(b) Also there exists a constant $M_A > 0$ such that, $\| A(t)g(t, x_t) \| \leq M_A \| x_t \|$, holds for $t \in I$

(H5). (a) For each $t \in I$, the function $k(t, s, .): C \to X$ is continuous and, for each $x \in C$, the function $k(., ., x): I \to X$ is strongly measurable.

(b) There exists an integrable function $m_k: I \to [0, \infty)$ such that, $\| k(t, s, x) \| \leq m_k(t)\Omega_1(\| x \|)$, holds for $t \in I, x \in C$, where $\Omega_1: [0, \infty) \to [0, \infty)$. is a continuous nondecreasing function.

(H6). (a) For each $t \in I$, the function $h(t, s, .): C \to X$ is continuous and, for each $x \in C$, the function $h(., ., x): I \to X$ is strongly measurable.

(b) There exists an integrable function $m_h: I \to [0, \infty)$ such that,

$$\| h(t, s, x) \| \leq m_h(t)\Omega_2(\| x \|),$$

holds for $t \in I, x \in C$, where $\Omega_2: [0, \infty) \to [0, \infty)$. is a continuous nondecreasing function.

(H7). The function $f: I \times C \times X \times X \to X$ satisfies the Caratheódory conditions:

(a) For each $t \in I$, the function $f(t, ., ., .): C \times X \times X \to X$ is continuous and, for each $(x, y, z) \in C \times X \times X$, the function $f(., x, y, z): I \to X$ is strongly measurable.

(b) There exists an integrable function $m_f: I \to [0, \infty)$ such that,

$$\| f(t, x, y, z) \| \leq m_f(t)\Omega_3(\| x \| + \| y \| + \| z \|),$$

holds for $t \in I, x \in C$, and $y, z \in X$, where $\Omega_3: [0, \infty) \to [0, \infty)$. is a continuous nondecreasing function.

(H8). The function $q: \mathcal{C}(I, X) \to X$ is continuous and there exists a constant $M_q \geq 0$ such that $\| q(x) \| \leq M_q$, for $x \in X$

(H9). The following inequality holds: The function

$$\widehat{m}(t) = max\{1, M_1 \| E^{-1} \| m_f(t), m_k(t), m_h(t), \int_0^t \frac{\partial}{\partial t} m_k(t)ds, \int_0^a \frac{\partial}{\partial t} m_k(t)ds\}$$

satisfies $\int_0^b \widehat{m}(s)ds < \int_d^\infty \frac{ds}{s + 2\Omega_1(s) + 2\Omega_2(s) + \Omega_3(s)}$,

where $d = \| E^{-1} \| M_1[\| E\phi(0) \| + \| EM_q \| + L_g]$

and

$$d_2 = M_2 \| E^{-1} \| \{\| x_1 \| + \| E^{-1} \| [M_g + M_1 L_g] + M_1 \| E^{-1} \| M_A bK$$
$$+ M_1 \| E^{-1} \| \int_0^b m_f(s)\Omega_3[K + \int_0^s m_k(\tau)\Omega(K)d\tau + \int_0^a m_h(\tau)\Omega_2(K)d\tau]\}$$

3 CONTROLLABILITY RESULT

Theorem 3.1 If the hypotheses $[H1] - [H9]$ are satisfied, then the system $(2.1) - (2.2)$ is controllable on I.

Proof. Using the hypothese $[H3]$ for an arbitrary function $x(.)$, define the control

$$\begin{aligned} u(t) = W^{-1}[x_1 - E^{-1}U(b, 0)[E\phi(0) - Eq(x) - g(0, x_0)] - E^{-1}g(b, x_b) \\ - \int_0^b E^{-1}U(b, s)A(s)g(s, x_s)ds \\ - \int_0^b E^{-1}U(b, s)f(s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau, \int_0^a h(s, \tau, x_\tau)d\tau)ds,](t) \end{aligned} \tag{3.1}$$

For $\phi \in C$, define $\hat{\phi} \in C_b$, $C_b = C([-r, b], X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t) - q(x), & -r \leq t \leq 0, \\ E^{-1}U(t, 0)[E\phi(0) - Eq(x)] & 0 \leq t \leq \text{b}. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r, b]$, it is easy to see that y satisfied

$y_0 = 0$

$$\begin{aligned} y(t) = E^{-1}g(t, y_t + \hat{\phi}_t) - E^{-1}U(t, 0)g(0, \hat{\phi}_0) + \int_0^t E^{-1}U(t, s)A(s)g(s, y_s + \hat{\phi}_s)ds \\ + \int_0^t E^{-1}U(t, \eta)CW^{-1}[x_1 - E^{-1}U(b, 0)[E\phi(0) - Eq(x) - g(0, \hat{\phi}_0)] \\ -E^{-1}g(b, y_b + \hat{\phi}_b) - \int_0^b E^{-1}U(b, s)A(s)g(s, y_s + \hat{\phi}_s)ds \end{aligned} \tag{3.2}$$

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$-\int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\ \int_0^a h(s,\tau,y_\tau+$
$\hat{\phi}_\tau)d\tau)ds,](\eta)d\eta\ +\int_0^t E^{-1}U(t,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+$
$\hat{\phi}_\tau)d\tau)ds,\ \ t\in I.$

if and only if x satisfies

$$\begin{aligned}x(t)&=E^{-1}U(t,0)[E\phi(0)-Eq(x)]+E^{-1}g(t,x_t)-E^{-1}U(t,0)g(0,x_0)\\&+\int_0^t E^{-1}U(t,s)A(s)g(s,x_s)ds\\&+\int_0^t E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,x_0)]\\&-E^{-1}g(b,x_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,x_s)ds\\&-\int_0^b E^{-1}U(b,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau)ds,](\eta)d\eta\\&+\int_0^t E^{-1}U(t,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau)ds,\quad t\in I.\end{aligned}$$

and $x(t)=\phi(t)-q(x),\quad t\in[-r,0].$

Define $C_b^0=\{y\in C_b: y_0=0\}$ and we now show that when using the control, the operator $F: C_b^0\to C_b^0$, defined by

$$(Fy)(t)=0,\qquad t\in[-r,0]$$

$$\begin{aligned}(Fy)(t)&=E^{-1}g(t,y_t+\hat{\phi}_t)-E^{-1}U(t,0)g(0,\hat{\phi}_0)+\int_0^t E^{-1}U(t,s)A(s)g(s,y_s+\hat{\phi}_s)ds\\&+\int_0^t E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,\hat{\phi}_0)]\\&-E^{-1}g(b,y_b+\hat{\phi}_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,y_s+\hat{\phi}_s)ds\\&-\int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\\&\hat{\phi}_\tau)d\tau)ds,](\eta)d\eta\ +\int_0^t E^{-1}U(t,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\\&\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,\quad t\in I.\end{aligned}$$

has a fixed point. This fixed point point is then a solution of equation (3.2).

Clearly $x(b)=x_1$ which means that the control u steers the system $(2.1)-(2.2)$ from the intial function ϕ to x_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

In order to study the controllability problem of $(2.1)-(2.2)$, we introduce a parameter $\lambda\in(0,1)$ and consider the following system

$$\frac{d}{dt}[Ex(t)-g(t,x(t))]=\lambda A(t)x(t)+\lambda Bu(t)+\lambda f(t,x(t),\int_0^t k(t,s,x(s))ds,\int_0^a h(t,s,x(s))ds)\quad t\in[0,b], \tag{3.3}$$

$$x(t)+q(x)=\lambda\phi(t)\quad t\in[-r,0] \tag{3.4}$$

First we obtain a priori boounds for the mild solution of the equation $(3.3)-(3.4)$. Then from

$$\begin{aligned}x(t)&=\lambda E^{-1}U(t,0)[E\phi(0)-Eq(x)]+\lambda E^{-1}g(t,x_t)-\lambda E^{-1}U(t,0)g(0,x_0)\\&+\lambda\int_0^t E^{-1}U(t,s)A(s)g(s,x_s)ds\\&+\lambda\int_0^t E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,x_0)]\\&-E^{-1}g(b,x_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,x_s)ds\\&-\int_0^b E^{-1}U(b,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau)ds,](\eta)d\eta\\&+\lambda\int_0^t E^{-1}U(t,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau)ds,\quad t\in I.\end{aligned}$$

we have

$$\begin{aligned}\|x(t)\|&\le\|E^{-1}U(t,0)[E\phi(0)-Eq(x)]\|+\|E^{-1}g(t,x_t)\|+\|E^{-1}U(t,0)g(0,x_0)\|\\&+\int_0^t\|E^{-1}U(t,s)A(s)g(s,x_s)\|ds\\&+\int_0^t\|E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,x_0)]\\&-E^{-1}g(b,x_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,x_s)ds\end{aligned}$$

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$$- \int_0^b E^{-1}U(b,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau)ds,](\eta) \parallel d\eta$$
$$+ \int_0^t \parallel E^{-1}U(t,s)f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau,\int_0^a h(s,\tau,x_\tau)d\tau) \parallel ds$$
$$\parallel x(t) \parallel \leq \parallel E^{-1} \parallel M_1[\parallel \phi(0) \parallel + M_q + \widehat{M}_q] + \parallel E^{-1} \parallel M_g K + M_1 \parallel E^{-1} \parallel \int_0^t M_A \parallel x_s \parallel ds + M_1 b d_2 + M_1 \parallel E^{-1} \parallel \int_0^t M_f(s)\Omega_3[\parallel x + \int_0^b M_k(\tau)\Omega_1(\parallel x \parallel) + \int_0^a M_h(\tau)\Omega_2(\parallel x \parallel)d\tau]ds$$

Let us take the right hand side of the above inequality as $\mu(t)$. Then we have $x(0) = \mu(0) = d$, and $\parallel x_(t) \parallel \leq \mu(t)$, with

$$\mu'(t) = M_1 \parallel E^{-1}M_A \parallel x_t \parallel + M_1 \parallel E^{-1} \parallel M_f(s)\Omega_3[\parallel x \parallel + \int_0^t M_k(s)\Omega_1(\parallel x \parallel)ds + \int_0^a M_h(s)\Omega_2(\parallel x \parallel)ds]$$
$$\leq d_1\mu(t) + M_1 \parallel E^{-1} \parallel M_f(s)\Omega_3[\mu(t) + \int_0^b M_k(s)\Omega_1(\mu(s))ds + \int_0^a M_h(s)\Omega_2(\mu(s))ds]$$

where $d_1 = M_1 \parallel E^{-1} \parallel M_A$. Since μ is obviously increasing, let

$$w(t) = \mu(t) + \int_0^t M_k(s)\Omega_1(\mu(s))ds + \int_0^a M_h(s)\Omega_2(\mu(s))ds$$

Then $w(0) = \mu(0) = c$ and $\mu(t) \leq w(t)$

$$w'(t) = \mu'(t) + M_k(t)\Omega_1(\mu(t)) + M_h(t)\Omega_2(\mu(t)) + \int_0^t \frac{\partial}{\partial s} M_k(s)\Omega_1(\mu(s))ds + \int_0^a \frac{\partial}{\partial s} M_h(s)\Omega_2(\mu(s))ds$$
$$\leq w(t) + M_1 \parallel E^{-1} \parallel M_f(t)\Omega_3(w(t)) + \int_0^t \frac{\partial}{\partial s} M_k(s)\Omega_1(\mu(s))ds + \int_0^a \frac{\partial}{\partial s} M_h(s)\Omega_2(\mu(s))ds$$
$$\leq \widehat{m}\{w(t) + \Omega_3(w(t)) + 2\Omega_1(w(t)) + 2\Omega_2(w(t))\}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega_3(s) + 2\Omega_1(s) + 2\Omega_2(s)} \leq \int_0^b \widehat{m}(s)ds \leq \int_d^\infty \frac{ds}{s + \Omega_3(s) + 2\Omega_1(s) + 2\Omega_2(s)}.$$

This inequality that there is a priori bound $K > 0$ such that $w(t) \leq K$ and hence $\mu(t) \leq K,\ t \in [0,b]$. Since $\parallel x(t) \parallel \leq K,\ t \in I$, we have

$$\parallel x \parallel_1 = sup\{\parallel x(t) \parallel : -r \leq t \leq b \parallel < K$$

where K depending only on b and the functions $M_f(.), M_k(.), M_h(.), \Omega_1(.), \Omega_2(.), \Omega_3(.)$

Next we must prove that the operator F is a completely continuous operator. Let $B_k = \{y \in C_b^0 : \parallel y \parallel_1 \leq K\}$ for some $K \geq 1$.

We first show that the set $\{Fy : y \in B_k\}$ is equicontinuous. Let $y \in B_k$ and $t_1, t_2 \in [0,b]$. Then if $0 < t_1 < t_2 \leq b$,

$$\parallel (Fy)(t_1) - (Fy)(t_2) \parallel$$
$$\leq \parallel E^{-1} \parallel \parallel g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2}) \parallel + \parallel E^{-1} \parallel \parallel U(t_1,0) - U(t_2,0) \parallel \parallel g(0,\hat{\phi}_0) \parallel$$
$$+ \parallel \int_0^{t_1} E^{-1}[U(t_1,s) - U(t_2,s)]A(s)g(s,y_s + \hat{\phi}_s)ds \parallel + \parallel \int_{t_1}^{t_2} E^{-1}U(t_2,s)A(s)g(s,y_s + \hat{\phi}_s)ds \parallel$$
$$+ \parallel \int_0^{t_1} E^{-1}[U(t_1,\eta) - U(t_2,\eta)]CW^{-1}[x_1 - E^{-1}U(b,0)[E\phi(0) - Eq(x) - g(0,\hat{\phi}_0)]$$
$$-E^{-1}g(b,y_b + \hat{\phi}_b) - \int_0^b E^{-1}U(b,s)A(s)g(s,y_s + \hat{\phi}_s)ds$$
$$- \int_0^b E^{-1}U(b,s)f(s,y_s + \hat{\phi}_s,\int_0^s k(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau)ds,](\eta)d\eta \parallel$$
$$+ \parallel \int_{t_1}^{t_2} E^{-1}U(t_2,\eta)CW^{-1}[x_1 - E^{-1}U(b,0)[E\phi(0) - Eq(x) - g(0,\hat{\phi}_0)]$$
$$-E^{-1}g(b,y_b + \hat{\phi}_b) - \int_0^b E^{-1}U(b,s)A(s)g(s,y_s + \hat{\phi}_s)ds$$

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$- \int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,](\eta)d\eta \parallel$
$+\parallel \int_0^{t_1} E^{-1}[U(t_1,s)-U(t_2,s)]f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds \parallel$
$+\parallel \int_{t_1}^{t_2} E^{-1}U(t_2,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds \parallel$
$\leq \parallel E^{-1} \parallel\parallel g(t_1,y_{t_1}+\hat{\phi}_{t_1})-g(t_2,y_{t_2}+\hat{\phi}_{t_2}) \parallel + \parallel E^{-1} \parallel\parallel U(t_1,0)-U(t_2,0) \parallel\parallel g(0,\hat{\phi}_0) \parallel$
$+\int_0^{t_1} \parallel E^{-1} \parallel\parallel [U(t_1,\epsilon)-U(t_2,\epsilon)]U(\epsilon,s)A(s)g(s,y_s+\hat{\phi}_s) \parallel ds + (t_2-t_1)M_1 \parallel E^{-1} \parallel M_A K'$
$+\int_0^{t_1} \parallel E^{-1} \parallel\parallel [U(t_1,\epsilon)-U(t_2,\epsilon)]U(\epsilon,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,\hat{\phi}_0)]$
$-E^{-1}g(b,y_b+\hat{\phi}_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,y_s+\hat{\phi}_s)ds$
$-\int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,](\eta) \parallel d\eta$
$+(t_2-t_1) \parallel E^{-1} \parallel M_1M_2[x_1+\parallel E^{-1} \parallel M_1\widehat{M}_g+\parallel E^{-1} \parallel M_gK'+M_1b \parallel E^{-1} \parallel M_AK'$
$+M_1b \parallel E^{-1} \parallel M_f(t)(K'+bM_kK'+bM_hK'),]$
$+\parallel \int_0^{t_1} E^{-1}[U(t_1,s)-U(t_2,s)]f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds \parallel$
$+(t_2-t_1) \parallel E^{-1} \parallel M_1M_f(t)(K'+bM_kK'+bM_hK')$

where $K'=K+\parallel \hat{\phi} \parallel$. The right hand side is independent of $y \in B_K$ and tends to zero as $t_2-t_1 \to 0$, since by the assumption $(H2)$ implies the continuity in the uniform operator topology.

Thus the set $\{Fy; y \in B_K\}$ is equicontinuous.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.
It is easy to see that the family FB_K is uniformly bounded. Next we show that $\overline{FB_K}$ is compact. Since we have shown that FB_K is an equicontinuous collection, it suffices, by the Arzela-Ascoli theorem, to show that F maps B_K into a precompact set in X.
Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_K$, we define

$$(F_\epsilon y)(t) = E^{-1}g(t,y_t+\hat{\phi}_t)-E^{-1}U(t,0)g(0,\hat{\phi}_0)+\int_0^{t_\epsilon} E^{-1}U(t,s)A(s)g(s,y_s+\hat{\phi}_s)ds$$
$$+\int_0^{t_\epsilon} E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,\hat{\phi}_0)]$$
$$-E^{-1}g(b,y_b+\hat{\phi}_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,y_s+\hat{\phi}_s)ds$$
$$-\int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,](\eta)d\eta + \int_0^{t_\epsilon} E^{-1}U(t,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,$$
$$\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,$$

Now, by the assumption $(H2)$, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t): y \in B_K\}$ is precom pact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $y \in B_K$ we have
$\parallel (Fy)(t)-(F_\epsilon y)(t) \parallel$

$$\leq \int_{t-\epsilon}^t \parallel E^{-1}U(t,s)A(s)g(s,y_s+\hat{\phi}_s) \parallel ds$$
$$+\int_{t-\epsilon}^t \parallel E^{-1}U(t,\eta)CW^{-1}[x_1-E^{-1}U(b,0)[E\phi(0)-Eq(x)-g(0,\hat{\phi}_0)]$$
$$-E^{-1}g(b,y_b+\hat{\phi}_b)-\int_0^b E^{-1}U(b,s)A(s)g(s,y_s+\hat{\phi}_s)ds$$
$$-\int_0^b E^{-1}U(b,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau)ds,](\eta) \parallel d\eta + \int_{t-\epsilon}^t \parallel E^{-1}U(t,s)f(s,y_s+\hat{\phi}_s,\int_0^s k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,$$
$$\int_0^a h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau) \parallel ds$$

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Therefore,

$$\| (Fy)(t) - (F_\epsilon y)(t) \| \to 0$$

as $\epsilon \to 0$, and there are precompact sets arbitrarily colse to the set $\{(Fy)(t): y \in B_K\}$. Hence the set $\{(Fy)(t): y \in B_K\}$ is precompact in X.

It remains to be shown that $F: C_b^0 \to C_b^0$ is continuous. Let $\{y_n \|_0^\infty \subset C_b^0$ with $y_n \to y$ in C_b^0. Then there is an integer K such that $\| y_n(t) \| \leq K$ for all n and $t \in I$, so $y_n \in B_K$ and $y \in B_K$. By $(H4), (H7)$, $g(t, y_n(t) + \hat{\phi}_t) \to g(t, y(t) + \hat{\phi}_t)$ for each $t \in I$ and since $\| g(t, y_n(t) + \hat{\phi}_t) - g(t, y_n(t) + \hat{\phi}_t) \| \leq 2M_g K'$, $A(t)g(t, y(t) + \hat{\phi}_t) \to A(t)g(t, y(t) + \hat{\phi}_t)$ for each $t \in I$ and since $\| A(t)g(t, y_n(t) + \hat{\phi}_t) - A(t)g(t, y(t) + \hat{\phi}_t) \| \leq 2M_A K'$, and $f(t, y_n(t) + \hat{\phi}_t, \int_0^t k(t,s,y_n(s) + \hat{\phi}_s)ds, \int_0^a h(t,s,y_n(s) + \hat{\phi}_s)ds) \to f(t, y(t) + \hat{\phi}_t, \int_0^t k(t,s,y(s) + \hat{\phi}_s)ds, \int_0^a h(t,s,y(s) + \hat{\phi}_s)ds)$ for each $t \in I$ and since $\| f(t, y_n(t) + \hat{\phi}_t, \int_0^t k(t,s,y_n(s) + \hat{\phi}_s)ds, \int_0^a h(t,s,y_n(s) + \hat{\phi}_s)ds) - f(t, y(t) + \hat{\phi}_t, \int_0^t k(t,s,y(s) + \hat{\phi}_s)ds, \int_0^a h(t,s,y(s) + \hat{\phi}_s)ds) \| \leq 2\mu_{K'}(t)$, $K' = K + \| \hat{\phi} \|$, we have, by dominated convergence theorem,

$$
\begin{aligned}
&\| (Fy_n)(t) - (Fy)(t) \| \\
&= sup_{t \in I} \| E^{-1}[g(t, y_n(t) + \hat{\phi}_t) - g(t, y(t) + \hat{\phi}_t)] \\
&\quad + \int_0^t E^{-1}U(t,s)[A(s)g(s, y_n(s) + \hat{\phi}_s) - A(s)g(s, y(s) + \hat{\phi}_s)]ds \\
&\quad + \int_0^t E^{-1}U(t,\eta)CW^{-1}[-E^{-1}[g(b, y_n(b) + \hat{\phi}_b) - g(b, y(b) + \hat{\phi}_b)] \\
&\quad - \int_0^b E^{-1}U(b,s)[A(s)g(s, y_n(s) + \hat{\phi}_s) - A(s)g(s, y(s) + \hat{\phi}_s)]ds \\
&\quad - \int_0^b E^{-1}U(b,s)[f(s, y_n(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau) \\
&\quad -f(s, y(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau)]ds,](\eta)d\eta \\
&\quad + \int_0^t E^{-1}U(t,s)[f(s, y_n(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau) \\
&\quad -f(s, y(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau)]ds \| \\
&\leq \| E^{-1} \| \| [g(t, y_n(t) + \hat{\phi}_t) - g(t, y(t) + \hat{\phi}_t)] \| \\
&\quad + M_1 \int_0^t \| E^{-1} \| [A(s)g(s, y_n(s) + \hat{\phi}_s) - A(s)g(s, y(s) + \hat{\phi}_s)]ds \\
&\quad + M_1 M_2 \int_0^t \| E^{-1} \| [\| E^{-1} \| \| [g(b, y_n(b) + \hat{\phi}_b) - g(b, y(b) + \hat{\phi}_b)] \| \\
&\quad + M_1 \int_0^b \| E^{-1} \| \| [A(s)g(s, y_n(s) + \hat{\phi}_s) - A(s)g(s, y(s) + \hat{\phi}_s)] \| ds \\
&\quad + M_1 \int_0^b \| E^{-1} \| \| [f(s, y_n(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau) \\
&\quad -f(s, y(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau)] \| ds](\eta)d\eta \\
&\quad + M_1 \int_0^t \| E^{-1} \| \| [f(s, y_n(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y_n(\tau) + \hat{\phi}_\tau)d\tau) \\
&\quad -f(s, y(s) + \hat{\phi}_s, \int_0^s k(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau, \int_0^a h(s,\tau,y(\tau) + \hat{\phi}_\tau)d\tau)] \| ds \\
&\to 0 \quad as \quad n \to \infty
\end{aligned}
$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0: y = \lambda F y, \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(F)$, the function $x = y + \hat{\phi}$ is a mild solution of $(3.3) - (3.4)$ for which we have proved that $\| x \|_1 \leq K$ and hence

$$\| y \|_1 \leq K + \| \hat{\phi} \|.$$

Consequently, by Schaefer's theorem, the operator F has a fixed point in C_0^b. This means that any fixed point of F is a mild solution of $(2.1) - (2.2)$ on I satisfying $(Fx)(t) = x(t)$. Thus the system $(2.1) - (2.2)$ is controllable on I.

Consider the Banach space $\mathcal{Z} = \mathcal{C}(J, X)$ with norm

$$\| x \| = \sup\{|x(t)|: t \in J\}.$$

We shall show that when using the control $u(t)$, the operator $\Psi: \mathcal{Z} \to \mathcal{Z}$ defined by

$$
\begin{aligned}
(\Psi x)(t) &= U(t,0)g(x) + \int_0^t U(t,s)f(s, x(s))ds \\
&+ \int_0^t U(t,s)BW^{-1}[x_1 - U(b,0)g(x) - \int_0^b U(b,s)f(s, x(\tau))d\tau](s)ds
\end{aligned}
$$

has a fixed point $x(\cdot)$. This fixed point is a mild solution of the system $(2.1) - (2.2)$ and this implies

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that the system is controllable on J.

Next consider the operators $v_1,\ v_2,\ v_3: \mathcal{C}(J,X) \to \mathcal{C}(J,X)$ defined by

$$(v_1 x)(t) = U(t,0)g(x),$$
$$(v_2 x)(t) = \int_0^t U(t,s)f(s,x(s))ds,$$
$$(v_3 x)(t) = \int_0^t U(t,s)BW^{-1}[x_1 - U(b,0)g(x) - \int_0^b U(b,\tau)f(\tau,x(\tau))d\tau](s)ds.$$

Lemma 3.1 Assume that $(H1)$ and $(H3)$ are satisfied and a set $Y \subset \mathcal{C}(J,X)$ is bounded. Then

$$\omega_0^t(v_2 Y) \le 2bN_1\beta(f([0,b]\times Y)), \quad for \quad t \in J.$$

Proof: Fix $t \in J$ and denote

$$Q = f([0,t]\times Y),$$

$$q^t(\epsilon) = \sup\{\| (U(t_2,s) - U(t_1,s))q \| : 0 \le s \le t_1 \le t_2 \le t,\ t_2 - t_1 \le \epsilon,\ q \in Q\}.$$

At the beginning, we show that

$$\lim_{\epsilon\to 0+} q^t(\epsilon) \le 2N_1\beta(Q). \tag{3.5}$$

Suppose contrary. Then there exists a number d such that

$$\lim_{\epsilon\to 0+} q^t(\epsilon) > d > 2N_1\beta(Q). \tag{3.6}$$

Fix $\delta > 0$ such that

$$\lim_{\epsilon\to 0+} q^t(\epsilon) > d + \delta > d > 2N_1(\beta(Q)+\delta). \tag{3.7}$$

Condition (3.8) yields that there exist sequences $\{t_{2,n}\},\{t_{1,n}\},\{s_n\} \subset J$ and $\{q_n\} \subset Q$ such that $t_{2,n} \to t',\ t_{1,n} \to t',\ s_n \to s$ and

$$\| (U(t_{2,n},s_n) - U(t_{1,n},s_n))q_n \| > d. \tag{3.8}$$

Let the points $z_1, z_2, \dots, z_k \in X$ be such that $Q \subset \bigcup_{i=1}^k B(z_i, \beta(Q)+\delta)$. Then there exists a point z_i and a subsequence $\{q_n\}$ such that $\{q_n\} \in B(z_i,\beta(Q)+\delta)$ that is,

$$\| z_j - q_n \| \le \beta(Q) + \delta,\ \ for \ \ n = 1,2,\dots$$

Further we obtain

$$\begin{aligned}
&\| U(t_{2,n},s_n)q_n - U(t_{1,n},s_n)q_n \| \\
&\le \| U(t_{2,n},s_n)q_n - U(t_{1,n},s_n)z_j \| + \| U(t_{2,n},s_n)z_j - U(t_{1,n},s_n)z_j \| \\
&\le N_1 \| q_n - z_j \| + N_1 \| z_j - q_n \| + \| U(t_{2,n},s_n)z_j - U(t_{1,n},s_n)z_j \| \\
&\le 2N_1(\beta(Q)+\delta) + \| U(t_{2,n},s_n)z_j - U(t_{1,n},s_n)z_j \|.
\end{aligned}$$

Letting $n \to \infty$ and using the properties of the evolution system $\{U(t,s)\}$ we get

$$\operatorname{limsup}_{n\to\infty} \| U(t_{2,n},s_n)q_n - U(t_{1,n},s_n)q_n \| \le 2N_1(\beta(Q)+\delta).$$

This contradicts (3.7) and (3.8).

Now fix $\epsilon > 0$ and $t_1,\ t_2 \in [0,t]$ such that $0 \le t_2 - t_1 \le \epsilon$. Applying $(H3)$, we get

$$\begin{aligned}
&\| (v_2 x)(t_2) - (v_2 x)t_1 \| \\
&\le \int_0^{t_1} \| \big(U(t_2,s) - U(t_1,s)\big)f\big(s,x(s)\big) \| ds + \int_{t_1}^{t_2} \| U(t_2,s)f\big(s,x(s)\big) \| ds \\
&+ \int_0^t \| (U(t_2,s) - U(t_1,s))f(s,x(s)) \| ds + \epsilon N_1 \sup\{\| f(s,x(s)) \| : x \in Y\}.
\end{aligned}$$

Hence we derive the following inequality

$$\begin{aligned}
\omega^t(v_1 Y,\epsilon) \quad &\le \sup\{\int_0^t \| (U(t_2,s) - U(t_1,s))f(s,x(s)) \| ds : t_1,t_2 \in [0,t],\ 0 \le t_2 - t_1 \le \epsilon, \\
&x \in Y\} + \epsilon N_1 \sup\{\| f(s,x(s)) \| : x \in Y\}.
\end{aligned}$$

Letting $\epsilon \to 0+$, we get the result.

Lemma 3.2 Assume that the assumptions $(H1),\ (H4)$ are satisfied and a set $Y \subset \mathcal{C}(J,X)$ is bounded. Then

$$\omega_0^t(v_1 Y) \le 2N_0(t)\beta(g(Y)), \quad for \quad t \in J.$$

The simple proof is omitted.

Lemma 3.3 Assume that the assumptions $(H1) - (H4)$ are satisfied and a set $Y \subset \mathcal{C}(J,X)$ is bounded. Then

$$\omega_0^t(v_3 Y) \le 2bN_1K_1(\| x_1 \| + N_0\beta(g(Y)) + bN_1\beta(f(Q))), \quad for \quad t \in J.$$

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Proof: As in the Lemmas 3.1 and 3.2, also fix $\epsilon > 0$ and $t_1,\ t_2 \in [0,t],\ 0 \leq t_2 - t_1 \leq \epsilon$. Applying $(H3)$ and $(H4)$, we get

$$\| (v_3x)(t_2) - (v_3x)(t_1) \|$$
$$\leq \int_0^{t_1} \| (U(t_2,s) - U(t_1,s))BW^{-1}[x_1 - U(b,0)g(x) - \int_0^b U(b,\tau)f(\tau,x(\tau))d\tau] \| ds$$
$$+ \int_{t_1}^{t_2} \| U(t_2,s))BW^{-1}[x_1 - U(b,0)g(x) - \int_0^b U(b,\tau)f(\tau,x(\tau))d\tau] \| ds$$
$$\leq K_1 \int_0^{t_1} \| U(t_2,s) - U(t_1,s) \| \Big[\| x_1 \| + \| U(b,0)g(x) \| + \int_0^b \| U(b,\tau)f\big(\tau,x(\tau)\big)d\tau \|\Big] ds + \epsilon K_1N_1[\| x_1 \| + N_0\sup\{\| g(x) \| : x \in Y\} + N_1\sup\{\| f(s,x(s)) \| : x \in Y\}].$$

Hence we derive the following inequality

$$\omega_0^t(v_3Y) \leq \sup\{K_1 \int_0^{t_1} \| U(t_2,s) - U(t_1,s) \| [\| x_1 \| + \| U(b,0)g(x) \|$$
$$+ \int_0^b \| U(b,\tau)f(\tau,x(\tau))d\tau \|]ds : t_1,t_2 \in [0,b],\ 0 \leq t_2 - t_1 \leq \epsilon, x \in Y\}$$
$$+ \epsilon K_1N_1[\| x_1 \| + N_0\sup\{\| g(x) \| : x \in Y\} + N_1\sup\{\| f(s,x(s)) \| : x \in Y\}].$$

Letting $\epsilon \to 0+$, we get

$$\omega_0^t(v_3Y) \leq 2bN_1K_1(\| x_1 \| + N_0\beta(g(Y)) + bN_1\beta(f(Q))).$$

Hence the proof.

Then we calculate our main result as follows:

Theorem 3.1 If the Banach space X is separable under the assumptions $(H1) - (H4)$, then system $(2.1) - (2.2)$ is controllable on J

Proof. Consider the operator $\mathcal{P}$ defined by

$$(\mathcal{P}x)(t) = U(t,0)g(x) + \int_0^t U(t,s)f(s,x(s))ds$$
$$+ \int_0^t U(t,s)BW^{-1}[x_1 - U(b,0)g(x) - \int_0^b U(b,s)f(s,x(\tau))d\tau](s)ds.$$

For an arbitrary $x \in C(J,X)$ and $t \in J$, we get

$$\| (\mathcal{P}x)(t) \| \leq N_0 \| g(x) \| + N_1 \int_0^t \| f(s,x(s)) \| ds$$
$$+ N_1K_1 \int_0^t [\| x_1 \| + N_0 \| g(x) \| + N_1 \int_0^b \| f(\tau,x(\tau)) \| d\tau]ds$$
$$\leq (1 + bN_1K_1)[N_0 \| g(x) \| + N_1 \int_0^b \| f(\tau,x(\tau)) \| d\tau] + bN_1K_1 \| x_1 \|.$$

From the above estimate and assumption $(H5)$ we infer that there exists a constant $r > 0$ such that the operator $\mathcal{P}$ transforms closed ball $\mathbb{B}$ into itself.

Now we prove that the operator $\mathcal{P}$ is continuous on $\mathbb{B}(\theta,r)$.

Let us fix $x \in \mathbb{B}(\theta,r)$ and take an arbitrary sequence $\{x_n\} \in \mathbb{B}(\theta,r)$ such that $x_n \to x$ in $C(J,X)$. Next we have

$$\| \mathcal{P}x_n - \mathcal{P}x \| \leq N_0 \| g(x_n) - g(x) \| + N_1 \int_0^t \| f(s,x_n(s)) - f(s,x(s)) \| ds$$
$$+ K_1 \int_0^t \| U(t,s) \| [N_0 \| g(x_n) - g(x) \|$$
$$+ N_1 \int_0^b \| f(\tau,x_n(\tau)) - f(\tau,x(\tau)) \| d\tau]ds$$
$$\leq (1 + bN_1K_1)[N_0 \| g(x_n) - g(x) \| + N_1 \int_0^b \| f(\tau,x_n(\tau)) - f(\tau,x(\tau)) \| d\tau].$$

Applying Lebesgue dominated convergence theorem, we derive that $\mathcal{P}$ is continuous on $\mathbb{B}(\theta,r)$.

Now we consider the sequence of sets $\{\Omega_n\}$ defined by induction as follows:

$$\Omega_0 = \mathbb{B}(\theta,r),\ \Omega_{n+1} = Conv(\mathcal{P}\Omega_n),\ for\ \ n = 1,2,\ldots.$$

This sequence is decreasing, that is, $\Omega_n \supset \Omega_{n+1},\ for\ \ n = 1,2,\ldots.$

Further let us put

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$$v_n(t) = \beta(\Omega_n([0,t])),$$
$$w_n(t) = \omega_0^t(\Omega_n).$$

Observe that each of the functions $v_n(t)$ and $w_n(t)$ is nondecreasing, while sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are non-increasing at any fixed $t \in J$. Put

$$v_\infty(t) = \lim_{n\to\infty} v_n(t),$$
$$w_\infty(t) = \lim_{n\to\infty} w_n(t), \ \ for \ \ t \in J.$$

Using Lemmas 2.2, 3.2 and $(H4)$, we obtain

$$\begin{aligned}\beta(v_1\Omega_n([0,t])) &\le \omega_0^t(v_1\Omega_n) + \sup_{s\le t}\beta(v_1\Omega_n(s))\\ &\le 2N_0(t)\beta(g(\Omega_n)) + \sup_{s\le t}N_0(s)\beta(g(\Omega_n))\\ &\le 3N_0(t)\beta(g(\Omega_n))\\ &\le 3m_gN_0(t)\beta(\Omega_n([0,b]))\\ &= 3m_gN_0(t)v_n(b),\end{aligned}$$

that is,

$$\beta(v_1\Omega_n([0,t])) \le 3m_gN_0(t)v_n(b). \tag{3.9}$$

Moreover

$$\begin{aligned}\beta(v_2\Omega_n([0,t])) &\le \omega_0^t(v_2\Omega_n) + \sup_{s\le t}\beta(v_2\Omega_n(s))\\ &\le 2bN_1(t)\beta(f([0,t]\times\Omega_n)) + \sup_{s\le t}\beta\left(\int_0^s U(s,\tau)f(\tau,\Omega_n(\tau))d\tau\right)\\ &\le 2m_fbN_1(t)\beta(\Omega_n([0,t])) + \sup_{s\le t}N_1(t)\int_0^s \beta(f(\tau,\Omega_n(\tau)))d\tau\\ &\le 2m_fbN_1(t)v_n(t) + m_fN_1(t)\int_0^t v_n(\tau)d\tau\end{aligned}$$

and

$$\begin{aligned}\beta(v_3\Omega_n([0,t])) &\le \omega_0^t(v_3\Omega_n) + \sup_{s\le t}\beta(v_3\Omega_n(s))\\ &\le 2bN_1(t)K_1(\| x_1 \| + N_0\beta(g(\Omega_n)) + bN_1\beta(f(Q)))\\ &\quad + \sup_{s\le t}\beta\{\int_0^t U(t,s)BW^{-1}[x_1 - U(b,0)g(\Omega_n) - \int_0^b U(b,\tau)f(\tau,\Omega_n(\tau))d\tau](s)ds\}\\ &\le 2bN_1(t)K_1(\| x_1 \| + N_0(t)\beta(g(\Omega_n)) + bN_1(t)\beta(f([0,t]\times\Omega_n)))\}\\ &\quad +\sup_{s\le t}bN_1(s)K_1\{\| x_1 \| + N_0\beta(g(\Omega_n)) + N_1(t)\int_0^s \beta(f(\tau,\Omega_n(\tau)))d\tau\}\\ &\le 3bN_1(t)K_1(\| x_1 \| + m_gN_0(t)v_n(b)) + bm_fN_1(t)K_1(2bN_1(t)v_n(t) + N_1\int_0^t v_n(\tau)d\tau).\end{aligned}$$

Linking this estimate with (3.9), we obtain

$$\begin{aligned}v_{n+1}(t) &= \beta(\Omega_{n+1}([0,t]))\\ &= \beta(\mathcal{P}\Omega_n([0,t]))\\ &\le \beta(v_1\Omega_n([0,t])) + \beta(v_2\Omega_n([0,t])) + \beta(v_3\Omega_n([0,t]))\\ &\le 3m_gN_0(t)v_n(b) + 2m_fbN_1(t)v_n(t) + m_fN_1(t)\int_0^t v_n(\tau)d\tau\\ &\quad + 3bN_1(t)K_1(\| x_1 \| + m_gN_0(t)v_n(b))\\ &\quad +bm_fN_1(t)K_1(2bN_1(t)v_n(t) + N_1\int_0^t v_n(\tau)d\tau).\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}v_\infty(t) &\le 3m_gN_0(t)v_\infty(b) + 2m_fbN_1(t)v_\infty(t) + m_fN_1(t)\int_0^t v_\infty(\tau)d\tau\\ &\quad + bm_fN_1(t)K_1(2bN_1(t)v_\infty(t) + N_1\int_0^t v_\infty(\tau)d\tau)\\ &\quad +3bm_gN_0(t)N_1(t)K_1v_\infty(b).\end{aligned}$$

Hence putting $t = b$, we get in view of $(H6)$

$$v_\infty(b) = 0. \tag{3.10}$$

Furthermore, applying Lemmas 3.1, 3.2, 3.3, we have

$$\begin{aligned}w_{n+1}(t) &= \omega_0^t(\Omega_{n+1})\\ &= \omega_0^t(\mathcal{P}\Omega_n)\\ &\le \omega_0^t(v_1\Omega_n) + \omega_0^t(v_2\Omega_n) + \omega_0^t(v_3\Omega_n)\end{aligned}$$

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$$\leq 2m_g N_0 v_n(b) + 2m_f b N_1 v_n(t) + 2bN_1K_1(\| x_1 \| + N_0\beta(g(Y)) + bN_1\beta(f(Q)))$$
$$\leq 2m_g N_0 v_n(b) + 2m_f b N_1 v_n(t) + 2bN_1K_1(\| x_1 \| + m_g N_0 v_n(b) + b m_f N_1 v_n(t))$$
$$\leq (2 + bN_1K_1)[m_g N_0 v_n(b) + m_f b N_1 v_n(t)].$$

Letting $n \to \infty$, we get

$$w_\infty(t) \leq (2 + bN_1K_1)[m_g N_0 v_\infty(b) + m_f b N_1 v_\infty(t)].$$

Putting $t = b$ and applying (3.10), we conclude that $w_\infty(b) = 0$. This fact together with (3.10) implies that $\lim_{n\to\infty}\chi(\Omega_n) = 0$. Hence, in view of the Remark 2.1, we deduce that the set $\Omega_\infty = \bigcap_{n=0}^{\infty} \Omega_n$ is nonempty, compact and convex. Finally. linking the above obtained facts concerning the set Ω_∞ and the operator $\mathcal{P}: \Omega_\infty \to \Omega_\infty$ and using the classical Schauder fixed point theorem, we infer that the operator $\mathcal{P}$ has at least one fixed point x in the set Ω_∞. Obviously the function $x = x(t)$ is a mild solution of $(2.1) - (2.2)$ satisfying $x(b) = x_1$. Hence the given system is controllable on J.

Remark 3.1 Let us consider the case when the mapping g is given by $g(x) = \sum_{i=1}^{n} d_i x(t_i)$, where $0 \leq t_1 < t_2 < \cdots < t_n \leq b$, $d_1, d_2, \ldots, d_n$ are given constants. For a bounded set $Y \subset \mathcal{C}(J, X)$ we get

$$\beta(g(Y)) \leq \sum_{i=1}^{n} |d_i| \beta(Y(t_i)) \leq \sum_{i=1}^{n} |d_i| \beta(Y(J)).$$

Similarly

$$\beta(g(Y)) \leq \sum_{i=1}^{n} |d_i| \beta(Y(t_i)) \leq \sum_{i=1}^{n} |d_i| . \sup_{t\in J} \beta(Y(t)).$$

These inequalities imply that the constant m_g from assumption $(H4)$ satisfies the following estimate $m_g \leq \sum_{i=1}^{n} |d_i|$.

Now let us consider the case, when the mapping g is given in the form $g(x) = \int_0^b h(t, x(t)) dt$,where the mapping $h: J \times X \to X$ satisfies the Carathéodory condition, and moreover $\beta(h(t, W)) \leq m(t)\beta(W)$ hold, for a.e. $t \in J$ and $W \subset X$, where the function $m: J \to \mathbb{R}+$ is integrable.

Then, for a bounded set $Y \subset \mathcal{C}(J, X)$, we have

$$\beta(g(Y)) \leq \beta(\int_0^b h(t, Y(t))dt) \leq \int_0^b m(t)\beta(Y(t))dt \leq \int_0^b m(t). \sup_{t\in J} \beta(Y(t))dt$$

and therefore the constant m_g from $(H4)$ satisfies the estimate

$$m_g \leq \int_0^b m(t).$$

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