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Mathematical Modeling of Subsurface Seepage Flow in 2-Dimensional Anisotropic Porous Medium

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Abstract

Prediction of water table fluctuation using mathematical models is an emerging technology. An analytical solution to predict the dynamic behavior of water table fluctuation in an anisotropic unconfined aquifer is presented in this paper. The aquifer overlying on leaky base is subjected to artificial recharge from top and withdrawal from wells. In 2-Dimensional aquifer number of basins and well is been considered. Two cases based on different hydrological systems is been analyzed. In the first case aquifer considered is having two side no-flow conditions and in the second case aquifer is isolated, having all four boundaries impervious. The seepage flow is approximated by non-linear partial differential equation called Boussinesq equation and is solved by finite Fourier transforms. The application of closed-form solution is demonstrated using an illustrative example. Effect of aquifer parameters on the formation of groundwater mound and cone of depression due to recharge and withdrawal are discussed. Effect of permeability of aquifer base is also observed.

Key words: *Leaky base , Boussinesq Equation, Recharges, Withdrawal , Fourier Transform*

Introduction:

In this new era mathematical models are the technique that can predict the transient as well as steady-state behavior of water table in unconfined aquifers under seepage and recharge conditions. These modeling techniques for simulation of subsurface seepage flow in confined as well as unconfined aquifers have been presented by several investigators like [20], [11],[18], [19], [26], [22], [1],[2],[3],[4],[5]. Some fundamental works concerning water table fluctuations in a rectangular-shaped homogeneous aquifer system due to localized recharge and withdrawal include [10], [99],[14],[15],[9], [17].

In this paper, a new analytical solution of a 2-dimensional linearized Boussinesq equation is developed. The hydrological setting of the model consists of an unconfined anisotropic aquifer overlaying a semi-pervious (leaky) base, subjected to recharge and withdrawal activities through multiple recharge basins and extraction/injection wells. This paper consists of a study of two hydrological systems. One is by considering two sides no flow condition and in the second case aquifer is isolated, having all four boundaries are impervious. The 2-dimensional linearized Boussinesq equation is solved using a finite Fourier transform. The closed-form solution is obtained. The sensitivity of the hydraulic head based on variation in aquifer parameters is analyzed. The result obtained in both cases is discussed and compared.

Development Of Mathematical Model

The model consists of an anisotropic unconfined aquifer of dimension A x B is underlain by a semi-pervious (leaky) base. Typically, such leaky semi-porous formations connect the unconfined aquifer with adjacent confined aquifers. Hydraulic conductivities of the aquifer along x and y directions are K_x and K_y respectively. The aquifer is in contact with two water bodies along with the coastlines $x = 0$ and $x = A$, maintaining a constant water head h_0 along these coastlines. The other two boundaries, namely $y = 0$ and $y = B$ of the aquifer are fully impervious similarly in the second case all four side impervious boundaries are considered. The initial depth is uniform h_0. Recharge and withdrawal activities are carried out using rectangular basins and point sized extraction wells located in the domain of the aquifer. If $h(x, y, t)$ denotes the variable water table measured from horizontal datum, then the groundwater flow in unconfined horizontal aquifer with semi-pervious base is governed by the following 2-dimensional partial differential equation:

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$$K_x \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + K_y \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) + P(x,y,t) = S\frac{\partial h}{\partial t} + \frac{k}{b}(h-h_0) \tag{1}$$

where S = specific yield; k and b are the hydraulic conductivity and thickness of the base. The term P(x, y, t) signifies the combined effects of recharge and withdrawal. The number of basins and wells are considered to be n_1 and n_2 respectively. The i^{th} basin is centered at (x_i, y_i) and is of dimension a_i x b_i, whereas the j^{th} well is located at (x_j, y_j). Recharge is considered at time-varying rate, whereas the extraction/injection is at a constant rate. Q_j is the rate of injection/extraction in the j^{th} well. δ is the Dirac delta function. Thus, define P(x, y, t)

$$P\left(x,\ y,\ t\right) = \left[\sum_{i=1}^{p_1} R_i\left(x,\ y,\ t\right) + \sum_{j=1}^{p_2} \omega_j\, Q_j\, \delta\left(x-x_j\right)\delta\left(y-y_j\right) \right] \tag{2}$$

where $R_i(x, y, t)$ denotes the transient recharge rate in the i^{th} basin extending from $x_i \le x \le x_i + a_i$; $y_i \le y \le y_i + b_i$. Assume that

$$R_i(x,y,t) = \begin{cases} N_{i0} + N_{i1}\, e^{-\lambda_i t}, & x_i \le x \le x_i + a_i\,;\ y_i \le y \le y_i + b_i \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where λ_i is a positive constant, determining the rate at which the recharge in the i^{th} basin reduces to a final value $N_{i\,0}$ from an initial value $N_{i\,0} + N_{i\,1}$.

The initial and boundary conditions are prescribed as follows:

For case I

$$h\left(x,y,t=0\right) = h_0,\ \left(\frac{\partial h}{\partial y}\right)_{y=0} = 0;\ \left(\frac{\partial h}{\partial y}\right)_{y=B} = 0, h\left(x=0,y,t\right) = h_0;\ \ h\left(x=A,y,t\right) = h_0 \tag{4}$$

For case II

$$h\left(x,y,t=0\right) = h_0, \left(\frac{\partial h}{\partial x}\right)_{x=0} = 0; \left(\frac{\partial h}{\partial x}\right)_{x=L} = 0,\ \left(\frac{\partial h}{\partial y}\right)_{y=0} = 0;\ \left(\frac{\partial h}{\partial y}\right)_{y=B} = 0 \tag{5}$$

Equation (1) is a second-order partial differential equation of parabolic nature, often referred to as a two-dimensional Boussinesq equation. Due to its nonlinearity, Boussinesq equation is analytically intractable. In order to find an approximate analytical solution of (1), we rewrite it in the form

$$K_x \frac{\partial^2 h^2}{\partial x^2} + K_y \frac{\partial^2 h^2}{\partial y^2} + 2P\left(x,y,t\right) = S\left(\frac{1}{h}\frac{\partial h^2}{\partial t}\right) + \frac{2k}{b}\frac{\left(h^2 - h_0^2\right)}{\left(h+h_0\right)} \tag{6}$$

Equation (7) is now linearized by the relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_t is the water head at the current moment (Marino 1973).

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{K_y}{K_x}\frac{\partial^2 h^2}{\partial y^2} + \frac{2}{K_x} P\left(x,\ y,\ t\right) = \frac{S}{K_x \hbar}\left(\frac{\partial h^2}{\partial t}\right) + \frac{k}{K_x b \hbar}\left(h^2 - {h_0}^2\right) \tag{7}$$

Now, define $H(x, y, t) = h^2 - h_0^2$, we get

$$\frac{\partial^2 H}{\partial x^2} + \frac{K_y}{K_x}\frac{\partial^2 H}{\partial y^2} + \frac{2}{K_x} P\left(x,\ y,\ t\right) = \frac{S}{K_x \hbar}\frac{\partial H}{\partial t} + \frac{k}{K_x b \hbar} H \tag{8}$$

The initial and boundary conditions read as

For case I

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$$H(x,\ y,\ 0)=0,\left(\frac{\partial H}{\partial y}\right)_{y=0}=0;\left(\frac{\partial H}{\partial y}\right)_{y=B}=0, H(x=0,\ y,\ t)=0;\quad H(x=A,\ y,\ t)=0 \tag{9}$$

For Case II

$$H(x,\ y,\ 0)=0,\ \left(\frac{\partial H}{\partial x}\right)_{x=0}=0;\ \left(\frac{\partial H}{\partial x}\right)_{x=L}=0,\ \left(\frac{\partial H}{\partial y}\right)_{y=0}=0;\ \left(\frac{\partial H}{\partial y}\right)_{y=B}=0 \tag{10}$$

Equation (8) along with the conditions (9) and (10) is solved using Fourier transform.

Thus, the solution of equation (8) is obtained as

For case I

$$\begin{aligned} h^2 = {h_0}^2 + \frac{8\nu}{ABK_x}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(\beta_m x)\cos(\gamma_n y)\Bigg[\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right) \\ +\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N_{i0}}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)+\frac{N_{i1}}{\alpha+\nu c-\lambda_i}\left(e^{-\lambda_i t}-e^{-(\alpha+\nu c)t}\right)\right\}\Bigg] \end{aligned} \tag{11}$$

For case II

$$\begin{aligned} h^2 = {h_0}^2 + \frac{8\nu}{ABK_x}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\beta_m x)\cos\left(\gamma_n\sqrt{\frac{Kx}{Ky}}\,y\right)\Bigg[\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right) \\ +\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N_{i0}}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)+\frac{N_{i1}}{\alpha+\nu c-\lambda_i}\left(e^{-\lambda_i t}-e^{-(\alpha+\nu c)t}\right)\right\}\Bigg] \end{aligned} \tag{12}$$

Discussion Of Results

The model domain is considered to have two rectangular basins, namely B_1 and B_2, each of dimension 10 m x 10 m, and centered at (35m, 35m) and (115m, 35m) respectively. Moreover, two extraction wells, namely W_1 and W_2, located at (35m, 75m) and (115m, 75m) are considered. Recharge is applied through B_1 and B_2 at a time-varying rate $0.5 + 0.8\ e^{-0.2t}$ and $0.4 + 0.9\ e^{-0.2t}$ per day respectively. At the same time, water is extracted from W_1 and W_2 at a constant rate 40 and 30 m^3/day respectively. Average saturated depth of the aquifer is determined using an iterative relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_t is the water head at the current moment (Marino 1973). Initial approximation of $\hbar$ is taken as h_0. Transient profiles of the water table fluctuations are determined for various values of time t.

Distribution of water head along line $y = 35\ m$ (line passing through the centers of recharge basins B_1 and B_2) for t = 1, 5 and 10 d is presented in Figure 1 and Figure 3.

It is observed that the groundwater mounds are symmetrical about the centers of the basins; however, the growth of mound beneath B_1 is marginally higher than that of beneath B_2, mainly due to varying recharge rate used in this example.

Hydraulic resistance of aquifer's base (measured by the ratio b/k) has significant impact on the transient profiles of the phreatic surface. Numerical experiments reveal that the groundwater mound attains a higher level in those aquifers which have comparatively higher values of hydraulic resistance. This is primarily due to vertical seepage loss through the aquifer's base, which decreases as the hydraulic resistance increases.

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Figure 1 Groundwater mound for k = 0.25 m/d

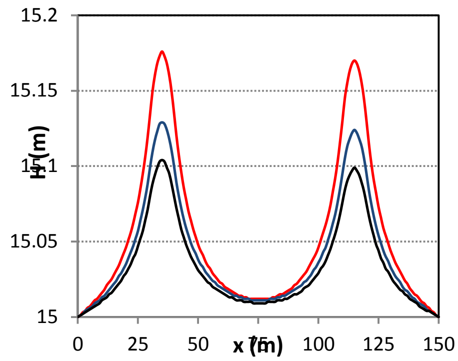


Figure.2 Cone of depression for k = 0.25 m/d

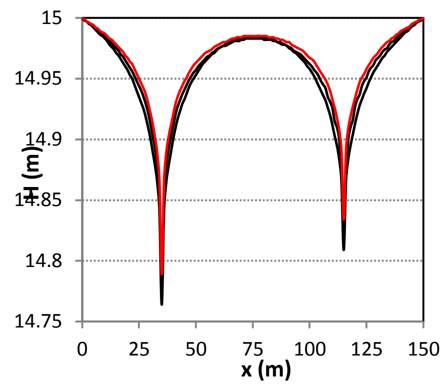


Figure 3:Ground water mound for t=1 and 5 day

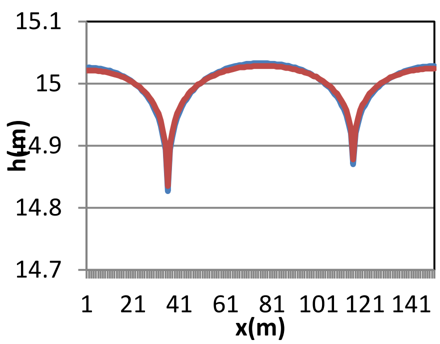
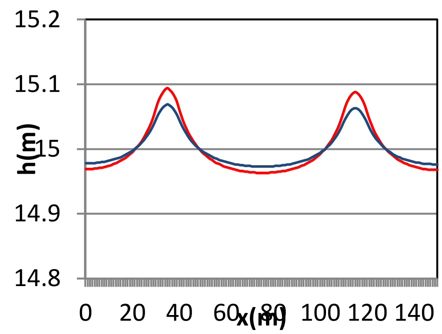


Figure 4:cone of depression for t=1 and 5 days



Lowering of water table due to continuous pumping from wells W_1 and W_2 are shown in Figure2 and Figure 4. These profiles characterize cones of depression at t = 1, 5 and 10 d in the presence of semi-pervious base with k = 0.25 m/d.

The difference in depth of cone under W_1 and W_2 is primarily due to varying pumping rate (Q_1 = 40 m^3/d, Q_2 = 30 m^3/d). It can be observed from these figures that the depth of cone increases with time. Moreover, water table depletion induced by pumping from wells is also affected by the hydraulic resistance of the aquifer's base. When the base is leaky, withdrawal from the wells is supplemented by the leakage induced vertical flow from hydraulically connected sources. Consequently, the depth of the cone of depression is mitigated. Three dimensional view of the groundwater mound and the cone of depression for t = 10 days is shown in Figure 5 (A and B) for k = 0.25 m/d and k = 0 in case I similarly in Figure 6(A and B) for case II.

Figure. 5 (A and B) Water table profile for k = 0.25 m/d and k = 0 in case I

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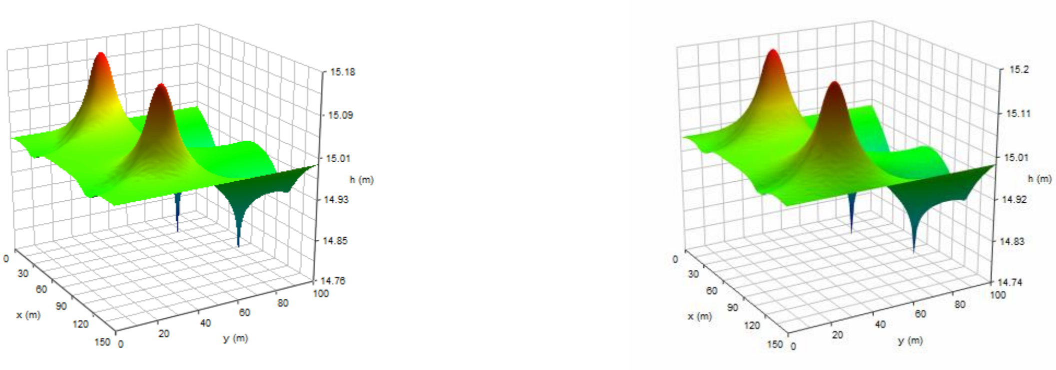
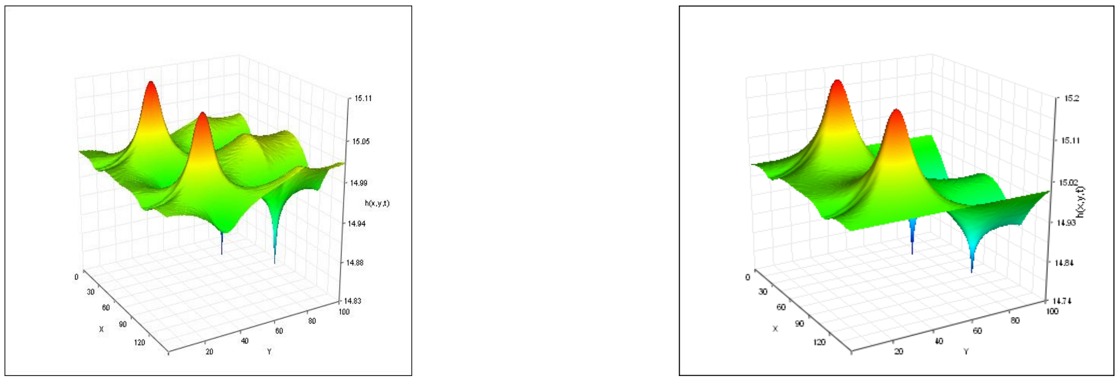


Fig. 6 (A and B) Water table profile for k = 0.25 m/d and k = 0 in case II



Conclusion:

In this paper, approximate analytical solution of 2-dimensional Boussinesq equation is developed to simulate water table fluctuations in a rectangular shaped unconfined aquifer due to multiple recharge and withdrawal is studied. The mathematical model consists of an anisotropic and homogeneous aquifer system overlying a leaky base, and hydraulically connected with two water bodies along its opposite faces. Two sides impervious and all four side impervious boundaries are considered. Analytical expressions for water head distribution are developed using finite Fourier transform. The conclusions obtained in this study are as follows:

- The solution developed in this study has the ability to predict the fluctuations in water table in unconfined aquifer due to multiple recharge and withdrawal.
- The numerical examples demonstrated that the semi-pervious layer supplements the draw-down beneath the wells, and reduces the height of groundwater mound beneath recharge basins.
- The cone of depression forms under the wells. Effect of water pumping from wells is clearly observed.
- The water mound form under recharge basin is broader in the case II that of the case I.
- Effect of semi-pervious base is clearly seen in the water mound and cone of depression.
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