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Mathematical Modeling of Subsurface Seepage Flow over Sloping Terrain due to Vertical Recharge and Seepage from Stream Stage Variations

Jyoti C Vanikar [b]
Rajeev K Bansal [a]
[b] *Research Scholar, JJT University, Chudela, Jhunjhunu, Rajasthan-333001 India*
[a] *Department of Mathematics, National Defence Academy, Khadakwasla, Pune-411023, India.*

Abstract.
This paper develops a mathematical model simulating time varying groundwater flow in an unconfined aquifer overlying a sloping impervious bed. The aquifer is contacted with two streams, one of which has a constant water level and the other stream is rising form an initial level to a final level by a known exponentially function of time. The aquifer is also receiving vertical recharge at a constant rate. The flow is simulated by a nonlinear Boussinesq equation and its approximate analytical solution is obtained using Laplace transform method. Closed form analytical expressions are derived for hydraulic head distribution in the aquifer and flow rate at the stream-aquifer interface. The results derived in the study are illustrated with a numerical example.
***Keywords**: Stream, Recharge, Aquifer, Laplace Transform, Boussinesq equation*

I. Introduction

Estimation of surface and groundwater interaction is an important aspect of hydrological investigation due to its key role in conjunctive management of groundwater resources. In the past, several analytical and numerical models have been developed to predict the surface-groundwater interaction in stream-aquifer systems under varying hydrological conditions [1-5].

The flow of groundwater in unconfined aquifers is usually formulated as a parabolic nonlinear Boussinesq equation which is not analytical tractable. So, most of the studies use a linearize version of the Boussinesq equation and develop analytical solution of the linearized equation. Though the analytical solution so developed provides useful insight in the flow process; the subsurface drainage over hillslope cannot be satisfactorily addressed with their results. Another perceived limitation is that they do not account for the gradual rise in the stream water.

The present study attempts to quantify the groundwater-surface water interaction when the adjacent stream rises gradually. The mathematical model developed here predicts the transient groundwater flow in a homogeneous unconfined aquifer of finite width overlying a downward sloping impervious bed owing to seepage from stream of varying water level and constant downward recharge. The aquifer is in contact with a constant water head at one end and a stream of time varying water level at another end. Furthermore, the aquifer is replenished by a constant recharge. Effect of bed slope, recharge rate and stream rise rate on the water table fluctuation and flow mechanism is analyzed using a numerical example.

II. Development of the Analytical Model and Analytical Solution

As shown in Fig.1, we consider an unconfined aquifer overlaying an impermeable sloping bed with downward slope $\tan\beta$. The aquifer is in contact with a constant piezometric level h_0 at its left end and a stream at the right end. The water in the stream is gradually rising from its initial level h_L to a final level h_0 by a known exponential decaying function of time t. Moreover, the aquifer is replenished vertically at a constant rate. If the variation in the hydraulic conductivity and specific yield of the aquifer with spatial coordinate is neglected, and the streamlines are considered to be nearly parallel to the impermeable bed (extended Dupuit-Forchheimer approach) then the groundwater flow in the aquifer can be characterized by the following nonlinear Boussinesq equation [13]

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - \tan\beta\frac{\partial h}{\partial x} + \frac{N\varepsilon(t)}{K\cos^2\beta} = \frac{\mathrm{S}}{\mathrm{K}\cos^2\beta}\frac{\partial h}{\partial t} \tag{1}$$

where $h(x,t)$ is the height of the water table measured above the impermeable sloping bed in the

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vertical direction. K and S respectively are the hydraulic conductivity and specific yield of the aquifer. N is the constant recharge rate. $\varepsilon(t)$ is a unit step function defined as follows:

$$\varepsilon(t) = \begin{cases} 0 \text{ if } t \leq 0 \\ 1 \text{ if } t > 0 \end{cases} \tag{2}$$

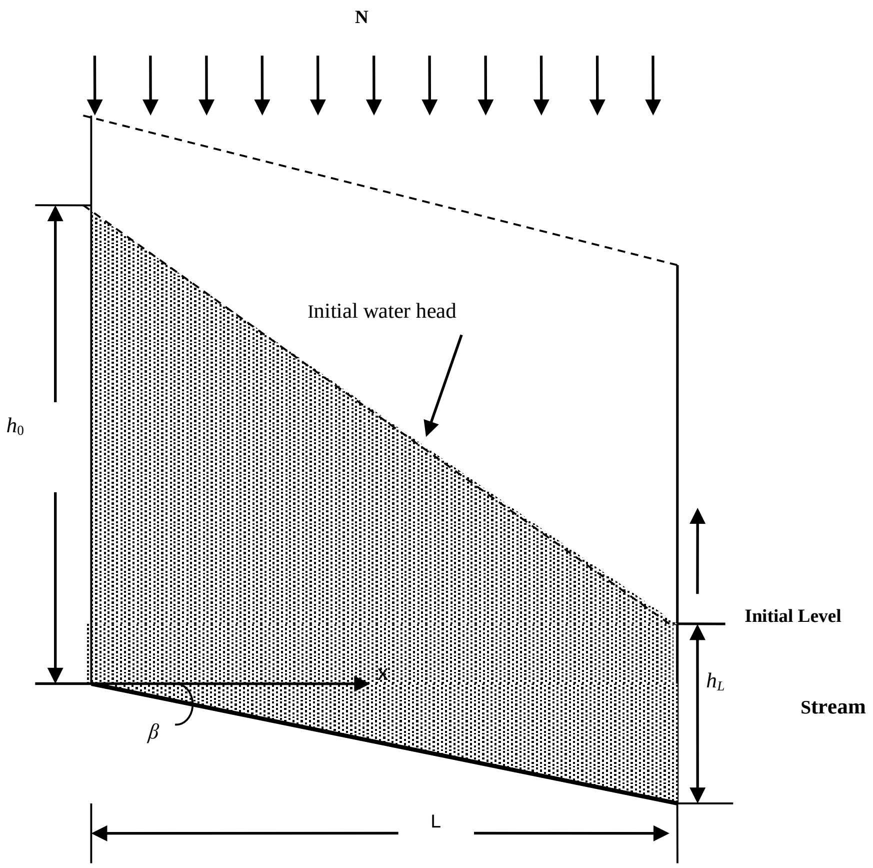

Fig. 1 Schematic diagram of an unconfined aquifer

The initial and boundary conditions are:

$$h(x, t=0) = h_0 + \frac{h_0 - h_L}{L}x \tag{3}$$

$$h(x=0, t) = h_0 \tag{4}$$

$$h(x=L, t) = h_0 - (h_0 - h_L)e^{-\lambda t} \tag{5}$$

where L is length of the aquifer. λ is a positive constant signifying the rate at which the water in the stream rises from its initial value h_L to a final level h_0. Equation (1) is a second order nonlinear parabolic partial differential equation which cannot be solved by analytical methods. However, an approximate analytical solution can be obtained by solving the corresponding linearized equation. In the present work, we adopt the linearization method as suggested by Marino [21]. Firstly, rewrite equation (1) as

$$\frac{\partial^2 h}{\partial x^2} - \frac{\tan\beta}{\overline{h}}\frac{\partial h}{\partial x} + \frac{N\varepsilon(t)}{K\,\overline{h}\cos^2\beta} = \frac{S}{K\overline{h}\cos^2\beta}\frac{\partial h}{\partial t} \tag{6}$$

where $\overline{h}$ is the mean saturated depth in the aquifer. The value of $\overline{h}$ is successively approximated using an iterative formula $\overline{h} = (h_0 + h_i)/2$, where h_0 is the initial water table height, and h_i is the varying

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water table height at time t at the end of which $\overline{h}$ is approximated. Equation (6) is further simplified using the following dimensionless variables and substitutions

$$H = \frac{h - h_0}{h_L - h_0}, \;\; X = \frac{x}{L}, \;\; \tau = \frac{K\,\overline{h}\cos^2\beta}{SL^2}t \tag{7}$$

Therefore, equation (6) becomes

$$\frac{\partial^2 H}{\partial X^2} - \frac{L\tan\beta}{\overline{h}}\frac{\partial H}{\partial X} + \frac{L^2 N\varepsilon(\tau)}{K\,\overline{h}(h_L - h_0)\cos^2\beta} = \frac{\partial H}{\partial \tau} \tag{8}$$

where

$$\varepsilon(\tau) = \begin{cases} 0 \text{ if } \tau \le 0 \\ 1 \text{ if } \tau > 0 \end{cases} \tag{9}$$

Now, define the following parameters

$$\alpha = \frac{L\tan\beta}{2\overline{h}}, N_1 = \frac{L^2 N}{K\overline{h}(h_L - h_0)\cos^2\beta}, \qquad \lambda_1 = \frac{SL^2}{K\overline{h}\cos^2\beta}\lambda \tag{10}$$

so that, equation (8) becomes

$$\frac{\partial^2 H}{\partial X^2} - 2\alpha\frac{\partial H}{\partial X} + N_1\varepsilon(\tau) = \frac{\partial H}{\partial \tau} \tag{11}$$

The initial and boundary conditions reduce to

$$H(X, \tau = 0) = X \tag{12}$$

$$H(X = 0, \tau) = 0 \tag{13}$$

$$H(X = 1, \tau) = 1 - e^{-\lambda_1 \tau} \tag{14}$$

Equation (11) along with the initial and boundary condition is solved using Laplace transform. The Laplace transform of $H(X,\tau)$ is defined as

$$L\{H(X,\tau): \tau \to s\} = \overline{H}(X,s) = \int_0^\infty H(X,\tau)e^{-s\tau}d\tau \tag{15}$$

The Laplace transform of equation (11) yields

$$\frac{\partial^2 \overline{H}}{\partial X^2} - 2\alpha\frac{\partial \overline{H}}{\partial X} + \frac{N_1}{s} = \frac{\partial \overline{H}}{\partial \tau} - X \tag{16}$$

The general solution of equation (16) can be found using elementary methods. Moreover, the arbitrary constants contained in the general solution are obtained by using the Laplace transform of equations (13)–(14) in it. We get

$$\begin{aligned}\overline{H}(X,s) = e^{\alpha X}\Bigg[&\left(\frac{2\alpha - N_1}{s^2}\right)\frac{\sinh\left((1-X)\sqrt{\alpha^2+s}\right)}{\sinh\left(\sqrt{\alpha^2+s}\right)} \\ &+ e^{-\alpha}\left(\frac{2\alpha - N_1}{s^2} - \frac{\lambda_1}{s(s+\lambda_1)}\right)\frac{\sinh\left(X\sqrt{\alpha^2+s}\right)}{\sinh\left(\sqrt{\alpha^2+s}\right)}\Bigg] + \frac{X}{s} \\ &- \frac{2\alpha}{s^2} + \frac{N_1}{s^2}\end{aligned} \tag{17}$$

Inverse Laplace transform of equation (17) is obtained from calculus of residue. Define

$$L^{-1}\{\overline{H}(X,s)\} = H(X,\tau) = \frac{1}{2\pi i}\lim_{w\to\infty}\int_{c-iw}^{c+iw}\overline{H}(X,s)e^{s\tau}ds \tag{18}$$

where c is an arbitrary positive number. The integration in equation (18) is performed along a line $s = c$ in the complex plane where $s = x + i\,y$. The real number c is chosen so that $s = c$ lies to the right of all singularities, but is otherwise arbitrary. The inverse Laplace transform of equation (17) yields

$$\begin{aligned} H(X,\tau) &= X - e^{-\alpha(1-X)}\left(1-e^{-\lambda_1\tau}\right)\frac{\sinh\left(X\sqrt{\alpha^2-\lambda_1}\right)}{\sinh\left(\sqrt{\alpha^2-\lambda_1}\right)} \\ &- 2\pi(2\alpha - N_1)e^{\alpha X}\left\{\sum_{n=1}^{\infty}\frac{n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)^2}\right. \\ &\left.- e^{-\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)^2}\right\} \\ &- 2\pi\lambda_1 e^{-\alpha(1-X)}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)(\alpha^2+n^2\pi^2-\lambda_1)} \end{aligned} \tag{19}$$

Equation (19) provides analytical expression for the water head distribution in the downward sloping aquifer under conditions mentioned in equations (3)–(5). One can obtain the corresponding results for an upward sloping by replacing α by – α and for a horizontal bed by setting $\alpha \to 0$. Furthermore, equation (19) can also be used to predict the water head profiles when the rise in the stream is extremely rapid; similar to the case of flood like situation, by letting $\lambda \to \infty$. We obtain

$$\begin{aligned} H(X,\tau) &= X - e^{-\alpha(1-X)}\frac{\sinh X}{\sinh 1} \\ &- 2\pi(2\alpha - N_1)e^{\alpha X}\left\{\sum_{n=1}^{\infty}\frac{n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)^2}\right. \\ &\left.- e^{-\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)^2}\right\} \\ &+ 2\pi e^{-\alpha(1-X)}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi X)\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2+n^2\pi^2)} \end{aligned} \tag{20}$$

III. Determination of Flow Rate and Steady State Profile

The flow rate $q(x,t)$ in the aquifer is defined as

$$q(x,t) = -Kh\left(\frac{\partial h}{\partial x} - \tan\beta\right) \tag{21}$$

and at the stream-aquifer interface, the flow rate is

$$q_{x=L} = -K\{h_0 - (h_0 - h_L)e^{-\lambda t}\}\left[\left(\frac{\partial h}{\partial x}\right)_{x=L} - \tan\beta\right] \tag{22}$$

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Invoking equation (20) in equation (22), the expressions for flow rate at the stream-aquifer interface are obtained as follows:

$$q_{x=L} = \frac{K}{L}\Bigg[1 - 2\pi^2(2\alpha - N_1)\left\{e^{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n n^2\left(1 - e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2 + n^2\pi^2)^2} - \sum_{n=1}^{\infty}\frac{n^2\left(1 - e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2 + n^2\pi^2)^2}\right\} - 2\pi^2\lambda_1\sum_{n=1}^{\infty}\frac{n^2\left(1 - e^{-(\alpha^2+n^2\pi^2)\tau}\right)}{(\alpha^2 + n^2\pi^2)(\alpha^2 + n^2\pi^2 - \lambda_1)} - \left(1 - e^{-\lambda_1\tau}\right)\left\{\alpha + \frac{\sqrt{\alpha^2 - \lambda_1}}{\tanh\sqrt{\alpha^2 - \lambda_1}}\right\} + \frac{L\tan\beta}{h_0 - h_L}\Bigg] \times (h_0 - h_L)\left\{h_0 + (h_L - h_0)e^{-\lambda\tau}\right\} \tag{23}$$

It is worth noting that despite continuous recharge, the profiles of water head attain a steady state value for large value of time. The expressions for steady-state water head and flow rate at stream-aquifer interface can be obtained by setting t → ∞ in equations (20) and (23), yielding

$$h^* = \frac{x}{L} - e^{-\alpha(1-x/L)}\frac{\sinh\left(\left(\frac{x}{L}\right)\sqrt{\alpha^2 - \lambda_1}\right)}{\sinh\left(\sqrt{\alpha^2 - \lambda_1}\right)} + 2\pi(2\alpha - N_1)e^{\alpha(x/L)}\left\{\sum_{n=1}^{\infty}\frac{n\sin(n\pi x/L)}{(\alpha^2 + n^2\pi^2)^2} - e^{-\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi x/L)}{(\alpha^2 + n^2\pi^2)^2}\right\} - 2\pi\lambda_1 e^{-\alpha(1-x/L)}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(n\pi x/L)}{(\alpha^2 + n^2\pi^2)(\alpha^2 + n^2\pi^2 - \lambda_1)} \times (h_0 - h_L)h_0 \tag{24}$$

$$q^* = \frac{K}{L}\Bigg[1 - 2\pi^2(2\alpha - N_1)\left\{e^{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{(\alpha^2 + n^2\pi^2)^2} - \sum_{n=1}^{\infty}\frac{n^2}{(\alpha^2 + n^2\pi^2)^2}\right\} - 2\pi^2\lambda_1\sum_{n=1}^{\infty}\frac{n^2}{(\alpha^2 + n^2\pi^2)(\alpha^2 + n^2\pi^2 - \lambda_1)} - \left\{\alpha + \frac{\sqrt{\alpha^2 - \lambda_1}}{\tanh\sqrt{\alpha^2 - \lambda_1}}\right\} + \frac{L\tan\beta}{h_0 - h_L}\Bigg] \times (h_0 - h_L)h_0 \tag{25}$$

IV. Numerical Solution of the Non-Linear Equation

To assess the efficiency of linearization technique, the non-linear Boussinesq equation is solved numerically by using Mac Cormack scheme. For this, equation (1) is written as

$$\frac{\partial h}{\partial t} = C_1 \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) - C_2 \frac{\partial h}{\partial x} + \frac{N}{S} \tag{26}$$

where, $C_1 = (K \cos^2 \beta)/S$ and $C_2 = (K \sin 2\beta)/2S$. Mac Cormack scheme is a predictor corrector scheme in which the predicted value of h is obtained by replacing the spatial and temporal derivatives by forward difference, i.e.

$$\begin{aligned} h^*_{k,n+1} = h_{k,n} + C_1 \frac{\Delta t}{(\Delta x)^2}\big[h_{k+1,n}\big(h_{k+1,n} - h_{k,n}\big) \\ - h_{k,n}\big(h_{k,n} - h_{k-1,n}\big)\big] - C_2 \frac{\Delta t}{\Delta x}\big(h_{k+1,n} - h_{k,n}\big) \\ + \frac{\mathrm{N}}{\mathrm{S}}\Delta \mathrm{t} \end{aligned} \tag{27}$$

The corrector is obtained by replacing the space derivative by backward differences, whereas the time derivative is still approximated by forward difference

$$\begin{aligned} h^{**}_{k,n+1} = h_{k,n} + C_1 \frac{\Delta t}{(\Delta x)^2}\big[h^*_{k,n+1}\big(h^*_{k+1,n+1} - h^*_{k,n+1}\big) \\ - h^*_{k-1,n+1}\big(h^*_{k,n+1} - h^*_{k-1,n+1}\big)\big] \\ - C_2 \frac{\Delta t}{\Delta x}\big(h^*_{k,n+1} - h^*_{k-1,n+1}\big) + \frac{\mathrm{N}}{\mathrm{S}}\Delta \mathrm{t} \end{aligned} \tag{28}$$

The final value of $h_{k,n+1}$ is given as an arithmetic mean of $h^*_{k,n+1}$ and $h^{**}_{k,n+1}$, i.e.

$$\begin{aligned} h_{k,n+1} = \frac{1}{2}\Big[h_{k,n} + h^*_{k,n+1} - C_2 \frac{\Delta t}{\Delta x}\big(h^*_{k,n+1} - h^*_{k-1,n+1}\big) \\ + C_1 \frac{\Delta t}{(\Delta x)^2}\big\{h^*_{k,n+1}\big(h^*_{k+1,n+1} - h^*_{k,n+1}\big) \\ - h^*_{k-1,n+1}\big(h^*_{k,n+1} - h^*_{k-1,n+1}\big)\big\}\Big] + \frac{\mathrm{N}}{\mathrm{S}}\Delta \mathrm{t} \end{aligned} \tag{29}$$

The initial and boundary conditions are descretized as

$$h_{k,1} = h_0 - \frac{h_0 - h_L}{L} x_k \tag{30}$$

$$h_{1,n+1} = h_0 \tag{31}$$

$$h_{L,n+1} = h_0 - (h_0 - h_L)e^{\lambda t_{n+1}} \tag{32}$$

Numerical experiments reveal that the method is stable if $\frac{A_1 \Delta t}{(\Delta x)^2} \le 0.06$ and $\frac{A_2 \Delta t}{\Delta x} \le 0.09$

V. Discussion of Results

To demonstrate the applicability of the closed form solution given by equations (19) and (23), we consider an aquifer with $\boldsymbol{L}$ = 150 m. Other hydrological parameters are: $\boldsymbol{K}$ = 3.6 m/h, $\boldsymbol{S}$ = 0.34, $\boldsymbol{h_0}$ = 5 m, $\boldsymbol{h_L}$ = 2 m, $\boldsymbol{N}$ = 4 mm/hr and $\boldsymbol{\lambda}$ = 0.054 per hr. Numerical experiments were carried out, and it is found that the first 50 terms of the summation series satisfactorily approximates the final value. The profiles of transient water head $\boldsymbol{h}$ obtained from equation (19) for $\boldsymbol{\beta}$ = 3 deg are plotted in Fig. 2 (continuous curves).

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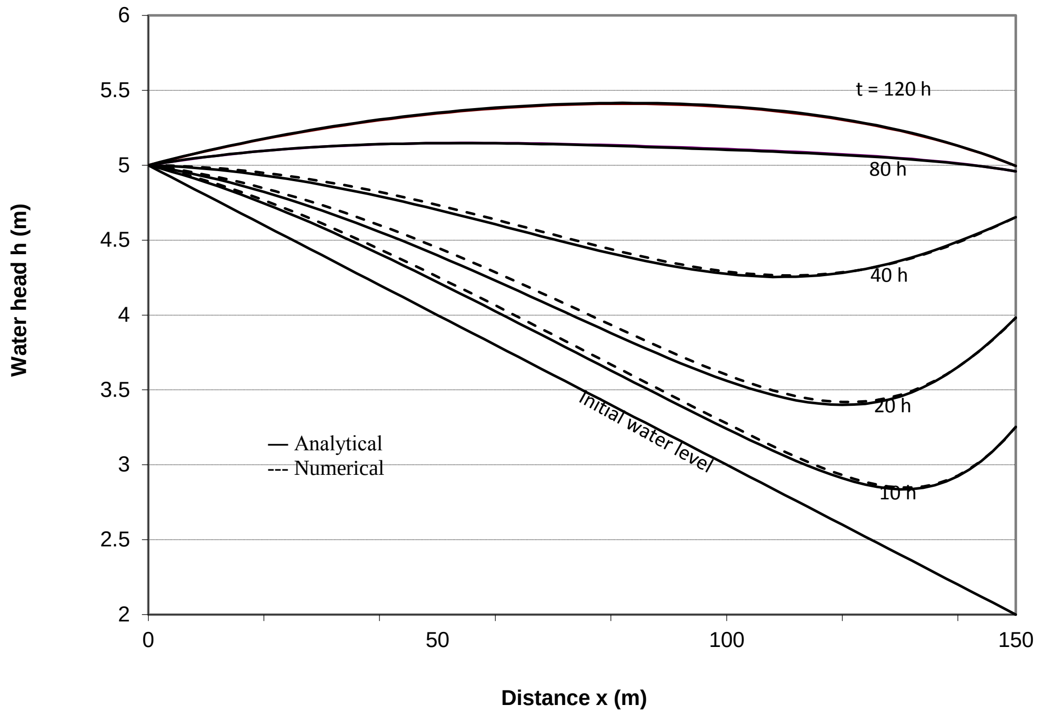

Fig. 2 Comparison of analytical and numerical solution for β = 3 deg

Numerical solution of the non-linear Boussinesq equation (1) using Mac Cormack scheme for the same data set is also presented in Fig. 2 (dotted curves). Close agreement between the numerical and analytical solutions demonstrates the efficiency of the linearization method adopted in this study.

VI. Conclusions

This study focuses on three major issues: (i) derive analytical expressions for water head and flow rate in an unconfined sloping aquifer due to continuous recharge and stream-varying water level, (ii) examine the efficiency and validity range of the linearization of method, and (ii) analyze the response of an aquifers to varying hydrological parameters. The linearized Boussinesq equation characterizing the transient groundwater flow is solved using Laplace transform technique, and the corresponding nonlinear equation is solved numerically by a fully explicit predictor-corrector scheme. Numerical experiments carried out in this study indicate that the evolution and stabilization of the phreatic surface and interaction of water between the stream and aquifer depend significantly on the bed slope and other aquifer parameters.

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