Matrix Representations for Sierpinski Graphs to Study Spectra at Different Stage of Iteration

Ashish Kumar*[1], Dr. Mohit James[2]

[1,2]Department of Mathematics and Statistics,SHUATS, Prayagraj, (U.P),India

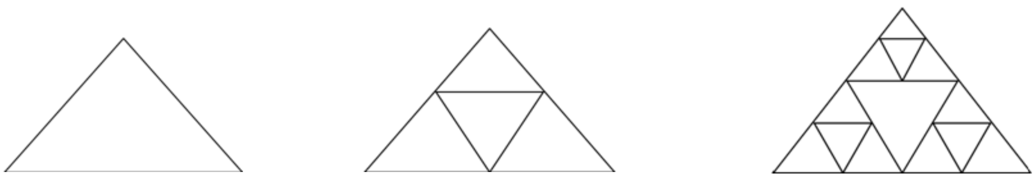
Abstract: Spectra of Sierpinski graph defined as eigenvalues of graph at different stage of iteration. In this is paper it aimed to found that the choice of matrix representation has a large impact, on the suitability of spectrum in a number of pattern recognition tasks, i.e. various stages of iteration.

Keyword: Signless Laplacian matrix, Spectra, Sierpinski Eulerian graph, Laplacian Matrix

Introduction:

Graph theory is a branch of mathematics started by Euler as early as 1736. A graph G consists of sets of vertices V and a set of edges E such that each edges is associated with an unordered pair of vertices then the graph is known as ***undirected graph*** and if each edges of graph is associated with an ordered pair of vertices then the graph is called directed graph or ***digraph***. Although graphs are frequently stored in a computer as list of vertices and edges, they are pictured as diagrams in the plane in a natural way. Vertex set of graph is represented as a set of points in a plane and edge is represented by a line segment or an arc (not necessarily straight).

We denote Sierpinski triangle by S_n that obtained at the n[th] stage of the iterative process.



Step 1: S_1 Step 2: S_2 Step 3: S_3

The ***generalised Sierpinski graph,*** as per the above definition of the Sierpinski graphs $S(n,k)$. The vertex set of $S(n,k)$ consists of all $n-$tuples of the integers (for every $n \geq 1$ and $k \geq 1$) i.e. $V(S(n,k)) = \{1,2,3..........,k\}^n$. Two different vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are adjacent if and only if there exists an $h \in \{1,2,...., n\}$ such that

(i) $u_t = v_t$, for $t = 1,2,....., h-1;$

(ii) $u_h \neq v_h$; and

(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,......, n.$

In 1736 Euler noticed that the river Pregel flows through the city of Konigsberg dividing the city into four land regions of which, two are banks and two are islands and the four land regions were connected by 7 bridges. Euler proposed that any given graph can be traversed with each edge traversed exactly once if and only if it had, zero or exactly two nodes with odd degrees. The graph following this condition is called, ***Eulerian circuit or path***. Exactly two nodes are, (and must be) starting and end of your trip. If it has even nodes than we can easily come and leave the node without repeating the edge twice or more. Using this theorem, we can create and solve number of problems.

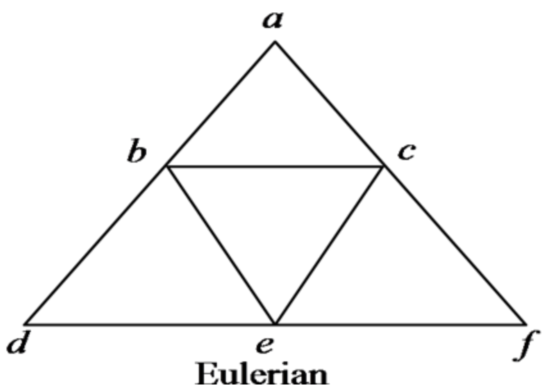
The existence of an Euler path in a graph is directly related to the degrees of the graph's vertices. Euler formulated the following theorems of which the first two set a sufficient and necessary condition for the existence of an Euler circuit or path in a graph respectively.

Theorem 1.2: An undirected graph has at least one Euler path if and only if it is connected and has two or zero vertices of odd degree.

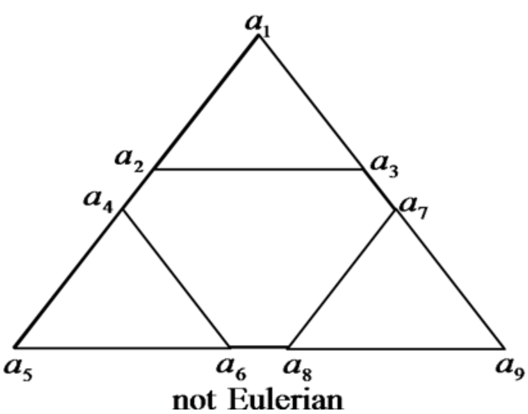
Theorem 1.3: An undirected graph has an Euler circuit if and only if it is connected and has zero vertices of odd degree.

Proposition 1: *Sierpinski's Gasket has an Euler circuit if and only if it is has two or zero vertices of odd degree.*

For the case of no odd vertices, the path can begin at any vertex and will end there; for the case of two odd vertices, the path must begin at one odd vertex and end at the other. Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.



Eulerian



not Eulerian

Proposition 2: *Sierpinski's Gasket is Eulerian if and only if its vertices are all of even degree.*

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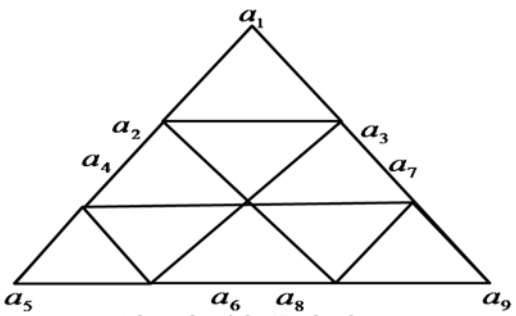
Proof:

***Case 1*(Eulerian as shown in figure):** Suppose *G* be a Sierpinski Graph is Eulerian, then G has an Eulerian trail which begins and ends at "*a*". If traverse along the trail then each and every time traverse a vertex having two edges. It is necessary condition that starting and ending nodes are same and each and every vertices must contain even degree (*deg(v)*) of vertices.

***Case 2*(not Eulerian as shown in figure):** Suppose *G* be a Sierpinski Graph is not Eulerian, then G has not Eulerian trail which begins at "a_1" but does not ends at "a_1". If traverse along the trail then each and every time traverse a vertex having two odd vertices or even vertices but above figure does not satisfy the Eulerian condition. Since each vertex in the middle of the trail is associated with three edges (G can not have just one odd vertex).

Let a_2, a_3, a_4, a_6, a_7 and a_8 be odd vertices in the connected graph G(not Eulerian). If we connect the vertices in pair $(a_2, a_8), (a_3, a_6)$ and (a_4, a_7) then the not Eulerian graph becomes the Sierpinski Eulerian. Hence all the vertices become even after connecting the odd vertices.



Sierpinski Eulerian

Eigenvalues of a graph

Let *A* be the adjacency matrix of the graph *Γ* of order *N*. Let *I* be the identity matrix of order *N*, and let *λ* be a scalar. Then the determinant **|*A*−*λI*|** which is an ordinary polynomial in λ of *N*-th degree with scalar coefficients, is called the characteristic polynomial of *Γ*. The roots of the equation |*A*−*λI*| = 0 are called the eigenvalues of the graph *Γ* (also of the matrix *A*). The set of eigenvalues is called the spectrum of the graph *Γ*. The multiplicity of an eigenvalue λ is called the algebraic multiplicity of *λ*. The equation ***Au = λu*** is called an eigenvalue equation. A nonzero solution u of the equation is called an eigenvector or eigenfunction for the eigenvalue *λ*. The vector space constructed from the set of eigenvectors corresponding to a particular eigenvalue *λ* is called the eigenspace of *λ*. The dimension of the eigenspace of an eigenvalue *λ* is the geometric multiplicity of *λ*. For a symmetric matrix, the geometric and algebraic multiplicities of an eigenvalue are equal.

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Laplacian Matrix:-

We consider graphs which has no loops or parallel edges, unless stated otherwise. Thus a graph $G = (V(G), E(G))$, consist of a finite set of vertices, $V(G)$, and a set of edges, $E(G)$, each of whose elements is a pair of distinct vertices. Given a graph, one associates a variety of matrices with the graph. Some of the important ones will be defined now. Let G be a graph with $V(G) = \{1,2,....., n\}, E(G) = \{e_1, e_2,....., e_n\}$. The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix with its rows and columns indexed by $V(G)$ and with the $(i, j)-$entry equal to 1 if vertices i, j are adjacent (*i.e.*, joined by an edge) 0(zero) otherwise. Thus $A(G)$ is a symmetric matrix with its $i-$th row (or column) sum equal to $d_i(G)$, which by definition is the degree of the vertex i, $i = 1,2,....., n.$ Let $D(G)$ denote the $n \times n$ diagonal matrix, $i-$th diagonal entry is $d_i(G), i = 1,2,....., n.$

The Laplacian matrix of G, denoted by $L(G)$, is simply the matrix $D(G) - A(G)$.

Signless Laplacian Matrix:-

The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix with its rows and columns indexed by $V(G)$ and with the $(i, j)-$entry equal to 1 if vertices i, j are adjacent (*i.e.*, joined by an edge) 0(zero) otherwise. Thus $A(G)$ is a symmetric matrix with its $i-$th row (or column) sum equal to $d_i(G)$, which by definition is the degree of the vertex i, $i = 1,2,....., n.$ Let $D(G)$ denote the $n \times n$ diagonal matrix, $i-$th diagonal entry is $d_i(G), i = 1,2,....., n.$

The Signless Laplacian matrix of G, denoted by $L(G)$, is simply the matrix $D(G) + A(G)$.

Theorem: Let *G* be a graph on n vertices with vertex degrees $d_1, d_2,....., d_n$ and largest *Q*-eigenvalue q_1. Then $2\min d_i \le q_1 \le 2\max d_i$. For a connected graph *G*, equality holds in either of these in equalities if and only if *G* is regular.

Theorem: Let *G* be a graph on n vertices with vertex degrees $d_1, d_2,....., d_n$ and largest *Q*-eigenvalue q_1. Then $\min(d_i + d_j) \le q_1 \le \max(d_i + d_j)$, where (*i,j*) runs over all pairs of adjacent vertices of *G*. For a connected graph *G*, equality holds in either of these inequalities if and only if *G* is regular or semi-regular bipartite.

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Proof: The line graph $L(G)$ of G has largest eigenvalue $q_1 - 2$. Consider an edge u of G which joins vertices i and j. The vertex u of $L(G)$ has degree $d_i + d_j - 2$. Hence, $\min(d_i + d_j - 2) \leq q_1 - 2 \leq \max(d_i + d_j - 2)$, which proves the theorem.

Lemma: Let $p(x)$ be a given polynomial. If λ is an eigenvalue of A, while x is an associated eigenvector, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$ and x is an eigenvector of $p(A)$ associated with $p(\lambda)$. The characteristic polynomial of A is defined by

$$\chi_A(t) = \det(tI - A)$$

Notes: The roots of the characteristic polynomial χ_A are exactly the eigenvalues of A. By the Fundamental Theorem of Algebra we know that every polynomial with degree n has exactly n complex roots (counted with multiplicities).

Lemma: Let A be a $n \times n-$matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then

$$\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$$

Lemma: Let A be a symmetric real matrix. Suppose v and w are eigenvectors of A associated with the eigenvalues λ and μ respectively. If $\lambda \neq \mu$ then $v \perp w$, i.e. v and w are orthogonal.

Proposition: The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.

Proof: Let $x^T = (x_1, ..., x_n)$.For a non-zero vector x we have $Qx=0$ if and only if $R^T x = 0$. The later holds if and only if $x_i = -x_j$ for every edge, i.e. if and only if G is bipartite. Since the graph is connected, x is determined up to a scalar multiple by the value of its coordinate corresponding to any fixed vertex i.

Theorem: (*Spectral Theorem*) Let A be a $n \times n$ symmetric real matrix. Then there are n pairwise orthogonal (real) eigenvectors v_i of A associated with real eigenvalues of A.

Consider $\lambda_1(A) \leq ... \leq \lambda_n(A)$ are eigenvalues of a symmetric matrix A. Some of these eigenvalues can be equal; we say that those eigenvalues have multiplicity greater than 1. Thus we denote the spectrum of A also in the form $\overline{\lambda}_1^{[m_1]}, ..., \overline{\lambda}_2^{[m_2]}$, where $\overline{\lambda}_i$ is an eigenvalue with multiplicity m_i.

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Theorem: (*Rayleigh-Ritz*) Let A be an $n \times n$ real symmetric matrix, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of A. Then

$$\lambda_n = \max_{x \neq 0} \frac{x^T Ax}{x^T x} = \max_{x^T x = 1} x^T Ax,$$

$$\lambda_1 = \min_{x \neq 0} \frac{x^T Ax}{x^T x} = \min_{x^T x = 1} x^T Ax.$$

Definiton: (Adjacency eigenvalues) The eigenvalues of $A(G)$ are called the adjacency eigenvalues of G. The set of all the adjacency eigenvalues are called the (adjacency) spectrum of the graph G.

Lemma[17,21]: Let G be a graph on n vertices.

i) The maximum eigenvalue of G lies between the average and the maximum degree of G, i.e.

$\bar{d} \leq \lambda_n \leq \Delta$.

ii) The range of all the eigenvalues of a graph is $-\Delta \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \Delta$.

Proof: i) The Rayleigh quotient for some special vector is greater than $\bar{d}$. This suffices to get the first inequality, because the maximum of the Rayleigh quotient is λ_n. The other inequality in (i) follows from the second point. Set $x = (1, 1, \ldots, 1)$T. The Rayleigh quotient for this vector equals:

$$R(A; x) = \frac{x^T Ax}{x^T x} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_{ij} x_j}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} \sum_{j: j \sim i} 1}{n} = \frac{\sum_{i=1}^{n} d_i}{n} = \bar{d}$$

ii) We have to show that the absolute value of every eigenvalue is less than or equal to the maximum degree. Let u be an eigenvector corresponding to the eigenvalue λ, and let u_j denote the entry with the largest absolute value. We have

$$|\lambda||u_j| = |\lambda u_j| = |(Au)_j| = \left| \sum_{i \sim j} u_i \right| \leq \sum_{i \sim j} |u_i| \leq d_j |u_j| \leq \Delta |u_j|.$$

Thus we have $|\lambda| \leq \Delta$ as required.

Definiton: (Laplacian eigenvalues) The eigenvalues of $L(G)$ are called the Laplacian eigenvalues. The set of all the Laplacian eigenvalues are called the (Laplacian) spectrum of the graph G.

Lemma[8]: Let G be a graph on n vertices with Laplacian eigenvalues $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and maximum degree Δ. Then $0 \leq \lambda_i \leq 2\Delta$ and $\lambda_n \geq \Delta$.

Proof: All eigenvalues are nonnegative by positive semidefinite matrices.

Let u be an eigenvector corresponding to the eigenvalue λ, and let u_j denote the entry with the largest absolute value. We have

$$|\lambda||u_j| = |\lambda u_j| = \left| d_j u_j - \sum_{i \sim j} u_i \right| \le d_j |u_j| + \sum_{i \sim j} |u_i| \le 2d_j |u_j| \le 2\Delta |u_j|.$$

Thus, we have $|\lambda| \le 2\Delta$ as required.

Let j be the vertex with maximal degree, i.e. $d_j = \Delta$. We define the characteristic vector x:

$$x_i = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Now, the desired inequality follows:

$$\lambda_n = \max_{\tilde{x} \ne 0} \frac{\tilde{x}^T \tilde{x}}{\tilde{x}^T \tilde{x}} \ge \frac{x^T L x}{x^T x} = \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{1} = \Delta.$$

We present the simple way MATLAB coding to generate the adjacency matrix, Laplacian matrix and signless Laplacian matrix of Sierpinski graph, hence to find its eigenvalues of given graph (i.e. *spectrum*).

MATLAB *code to generate the matrix representation of a Sierpinski graph and Sierpinski Eulerian graph on n vertices:*

```
%%construct of adjacency matrix on n vertices
nrows = n;
ncols = n;
A = ones(nrows,ncols);
for c = 1:ncols
    for r = 1:nrows

        if r == c
            A(r,c) = 0;
            A(n,n+1)=1;
        elseif abs(r-c) == 0
            A(r,c) = 1;
        else
            A(r,c) = 0;
        end

    end
end
A
%% construct eigenvalue of Sierpinski and Sierpinski Eulerian graph on n
vertices
eigenvalueofadjmatrix=eig(A)
%% create diagonal matrix
v=[d1 d2 d3 . . . . dn];
D=diag(v)
%% Laplacian matrix of Sierpinski graph and Sierpinski Eulerian graph
L=D-A
```


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```
%% create eigenvalues of Laplacian matrix of Sierpinski and Sierpinski
Eulerian graph  on n vertices
eigenvalueoflapmatrix=eig(L)
%% signless Laplacian matrix of Sierpinski and Sierpinski Eulerian graph
L'=D+A
%% create eigenvalue of signless Laplacian matrix of Sierpinski and
Sierpinski Eulerian graph on n vertices
eigenvalueofsiglapmatrix=eig(L')
%% graphical comparison between matrix of Sierpinski and Sierpinski
Eulerian graph on n vertices
y2 = ['enter all x coordinates'];
y1 = ['enter all x coordinates'];
xlabel(y1)
xlabel(y2)
plot(y2,'g--*')
hold;
plot(y1,'r--*')
```

Results:

Spectrum (eigenvalues) of 2nd stage of iteration on 3 vertices (i.e. *S*(2,3)) round off to 2 decimal places

	Sierpinski graph		Sierpinski Eulerian graph	
Sr. No.	**Laplacian matrix**	**Signless Laplacian Matrix**	**Laplacian matrix**	**Signless Laplacian Matrix**
1	0.00	1.00	0.00	1
2	0.70	1.00	1.27	1.27
3	0.70	1.00	1.27	1.27
4	3.00	1.44	3.00	1.63
5	3.00	2.38	4.00	4.00
6	3.00	2.38	4.00	4.00
7	4.30	4.62	4.73	4.73
8	4.30	4.62	4.73	4.73
9	5.00	5.56	7	7.73

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Spectrum (eigenvalues) of 3rd stage of iteration on 3 vertices (i.e. $S(3,3)$) round off to 2 decimal places

Sierpinski graph			Sierpinski Eulerian graph	
Sr. No.	**Laplacian matrix**	**Signless Laplacian Matrix**	**Laplacian matrix**	**Signless Laplacian Matrix**
1	0.00	1.00	0	7.87
2	0.14	1.00	0.33	7.37
3	0.14	1.00	1.29	5.68
4	0.70	1.00	6.83	7.42
5	0.70	1.00	0.30	4.82
6	0.70	1.00	1.71	3.89
7	1.10	1.09	5.58	2.43
8	1.10	1.30	3.05	1.14
9	1.38	1.30	4.53	1.43
10	3.00	1.87	4.53	1.36
11	3.00	1.87	4.14	1.26
12	3.00	2.20	6.73	1.35
13	3.00	2.38	1.72	2.40
14	3.00	2.38	5.72	4.83
15	3.00	2.38	3.27	4.73
16	3.62	3.00	4.28	6.00
17	3.90	3.00	7.00	6.00
18	3.90	3.00	7.00	1.00
19	4.30	4.62	2.00	1.00
20	4.30	4.62	2.00	3.00
21	4.30	4.62	4.00	3.00
22	4.86	4.80	4.00	4.00

23	4.86	5.12	5.00	4.00
24	5.00	5.12	5.00	4.00
25	5.00	5.70	4.00	4.00
26	5.00	5.70	4.00	4.00
27	5.00	5.91	4.00	4.00

After manipulating Sierpinski graph which is not Eulerian into Sierpinski Eulerian graph by introducing an edge between two vertices that having odd number of degrees.

From above given data, it is found that the spectrum (i.e. eigenvalue) of graph is used to characteristically encode the properties graphs. This graph model of a system provides a powerful means for this purpose. Likely, how can graph behave when it is at different stage of iterations (i.e. study of topological behaviour, structure, networking etc.). Keeping all in mind we can say that on comparison to spectrum of Sierpinski graph and Sierpinski Eulerian graph using matrix representations it can be stated that the Signless Laplacian spectrum has more representational power than the Laplacian spectrum, in terms of resulting of above graphs.

Conclusion:

This is a strong basis for believing that almost all graphs are determined by their spectra when "*n*" tends towards the infinity. We hope to have increased awareness about the importance of the choice of representation matrix for graph signal processing applications and other fields of computer science, science, mathematics and other aspects of NP problems.

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