

Congruency between classical and mild solutions of Caputo fractional impulsive evolution equation on Banach Space

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Abstract

This article derives a set of sufficient conditions for existence and uniqueness of mild and classical solutions using fixed point theorem. It also derives the sufficient conditions in which classical and mild solutions are congruent to each other.

Keywords- Existence of solution, Fractional evolution equation, Fixed point theorem, Impulsive conditions.

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1 Introduction

This article derives sufficient conditions for existence and uniqueness of mild and classical solutions of fractional evolution equation

$$\begin{aligned} {}^c D^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \quad k = 1, 2, \dots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \dots, p \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

over the interval $[0, T_0]$ in a Banach space X . Here A is bounded linear operator on X and $f : [0, T] \times X \times X \times X \rightarrow X, T, S : X \rightarrow X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(s))ds$. Where $h : D_0 \times X \rightarrow X, D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \rightarrow X, D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are continuous. Mathematical problems having such nonlinear equations arise in many physical situations like heat flow in materials with viscoelastic behavior [1].

Now a days, many physical phenomena like seepage flow in porous media [2], anomalous diffusion, wave and transport [3, 4, 5] and other problems in fluid dynamics [6] need modern mathematical modeling and solutions. In fact fractional differential equations are considered as an alternative model for nonlinear model [15]. This is because of their non local property unlike integer order differential equations [16] which means, that the next state of the system depends not only upon

its current state but also upon its entire historical states.

On the other-hand due to sudden change at certain moments in dynamics of the systems, such systems are modeled in to impulsive differential equations [21]. The existence and uniqueness of integer order impulsive system for many systems are studied by researchers like Gao, Liu, Rogovchenko, Anguraj and Arjunan [22] using various conditions.

Existence and uniqueness of mild solutions of impulsive fractional differential equations with classical conditions using transition matrix and semi-group theory are studied by Benchohra [23], Mophou [25, 24] and Ravichandran and Arjunan [26]. However, both the Riemann-Liouville and the Caputo fractional differential operators do not possess semi-group or commutative properties, which are inherent to the derivatives on integer order [27, 28, 29]. Therefore, there should be another approach to study fractional differential equation. The existence and uniqueness of the mild solution of impulsive fractional order evolution equation with initial classical and nonlocal conditions studied by [28]. Balachandran et. al.[29] studied existence and uniqueness of mild solution of impulsive fractional differential equations with delay using Banach fixed point theorem. Wang et. al. [30, 36] studied existence and uniqueness of the solutions of fractional order impulsive differential equation with order lies between 1 and 2 using fixed point theorem. Recently, Kataria and Patel [31] studied existence and uniqueness of mild solution of fractional integro-differential equation. However above authors discussed existence and uniqueness of mild solution. The authors of this article extended the work of them and discusses sufficient conditions for existence and uniqueness of classical solutions and conditions for which mild solution becomes classical solution.

The outline of the article is as follows: preliminaries are discussed in section 2, section 3 discusses sufficient conditions for existence and uniqueness of mild solution of (1.1), section 4 derives conditions for which mild solution give rise to classical solution.

2 Preliminaries

Some basic definitions and properties of fractional calculus and fractional differential equations used in this article, are as follows:

Definition 2.1. *The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I_{t_0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds,$$

provided the integral on right side exist. Where $\Gamma(\cdot)$ is gamma function [27].

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as*

$$D_{t_0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$ [27].

This derivative has singularity at zero and also requires special initial condition which lacks physical interpretation. To overcome this difficulty, Caputo [32] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$${}^c D_{t_0+}^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} \frac{d^n f(s)}{ds^n} ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n - 1)$ [28].

Fractional integral and differential operator satisfy following properties which is mentioned in Kilbas *et. al.* [33] and Samko *et. al.* [34].

Theorem 2.1. For $\alpha, \beta > 0$ and f having absolutely continuous derivatives up to suitable order, then

- (1) $I_{t_0+}^\alpha I_{t_0+}^\beta f(t) = I_{t_0+}^{\alpha+\beta} f(t)$
- (2) $I_{t_0+}^\alpha I_{t_0+}^\beta f(t) = I_{t_0+}^\beta I_{t_0+}^\alpha f(t)$
- (3) $I_{t_0+}^\alpha (f(t) + g(t)) = I_{t_0+}^\alpha f(t) + I_{t_0+}^\alpha g(t)$
- (4) $I_{t_0+}^\alpha {}^c D_{t_0+}^\alpha f(t) = f(t) - f(0)$, $0 < \alpha < 1$
- (5) ${}^c D_{t_0+}^\alpha I_{t_0+}^\alpha f(t) = f(t)$
- (6) ${}^c D_{t_0+}^\alpha f(t) = I_{t_0+}^{1-\alpha} f'(t)$, $0 < \alpha < 1$
- (7) ${}^c D_{t_0+}^\alpha {}^c D_{t_0+}^\beta f(t) \neq {}^c D_{t_0+}^{\alpha+\beta} f(t)$
- (8) ${}^c D_{t_0+}^\alpha {}^c D_{t_0+}^\beta f(t) \neq {}^c D_{t_0+}^\beta {}^c D_{t_0+}^\alpha f(t)$

For convenience, ${}^c D_{0+}^\alpha$ is taken as ${}^c D^\alpha$.

2.1 Notations

(N1) $X =$ Banach space and $D(A) =$ Domain of an operator A .

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0, T_0], X) = \{x : [0, T_0] \rightarrow X/x \text{ is continuous}\}$ with norm $\|x\| = \sup_t \|x(t)\|$

(N4) $PC([0, T_0], X) = \text{Closure} \left(\left\{ x : [0, T_0] \rightarrow X; x \in C([t_{k-1}, t_k], X), \text{ and } x(t_k^-) \text{ and } x(t_k^+) \text{ exist, } k = 1, 2, \dots, p \text{ with } x(t_k^-) = x(t_k) \right\} \right)$ with norm $\|x\|_{PC} = \sup_{t \in [0, T_0]} \|x(t)\|$

(N5) $AC([0, T_0], X) = \{x : [0, T_0] \rightarrow X/x \text{ is absolutely continuous}\}$ with norm $\|x\| = \sup_t \|x(t)\|$

(N6) $B(X) = \{A : X \rightarrow X/A \text{ is bounded and linear}\}$ with norm $\|A\|_{B(X)} = \sup\{\|A(y)\|; y \in X \& \|y\| \leq 1\}$

(N7) $C^\alpha([0, T_0], X) = \{x : [0, T_0] \rightarrow X/{}^c D^\alpha x(t) \text{ exist and continuous at each } t \in [0, T_0]\}$

With these definitions and properties, sufficient conditions for existence and uniqueness of solutions are derived as follows:

3 Mild Solution

In this section sufficient conditions are derived for existence and uniqueness of mild solution of the equation (1.1).

Definition 3.1. A function $x(t) \in PC([0, T_0], X)$ is a mild solution of the equation (1.1) if it satisfies

$$\begin{aligned} x(t) = & x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s)x(s)ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0 < t_k < t} I_k x(t_k^-). \end{aligned} \quad (3.1)$$

The definition of mild solution defined by authors is justified [36].

3.1 Assumptions

- (H1) $A(t) : X \rightarrow X$ is continuous bounded linear operator and there exists a positive constant M , such that $\|A(t)x - A(t)y\|_{B(X)} \leq M\|x - y\|$, for all $x, y \in X$.
- (H2) $f : [0, T_0] \times X \times X \times X \rightarrow X$ is continuous and there exists positive constants L_1, L_2 and L_3 , such that $\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|$ for all x_1, x_2, x_3, y_1, y_2 and y_3 in X .
- (H3) $h : D_0 \times X \rightarrow X$ and $k : D_1 \times X \rightarrow X$ are continuous and there exists positive constants H and K , such that $\|h(t, s, x) - h(t, s, y)\| \leq H\|x - y\|$ and $\|k(t, s, x) - k(t, s, y)\| \leq K\|x - y\|$ for all x and y in X .
- (H4) The functions $I_k : X \rightarrow X$ are continuous and there exist positive constants I_k^* for all $k = 1, 2, \dots, p$, such that $\|I_k x - I_k y\| \leq I_k^*\|x - y\|$ for all x and y in X .

Set, $\gamma = \frac{T_0^\alpha}{\Gamma(\alpha+1)}$ and further assume that,

$$(H5) \quad q = \left\{ \gamma[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3] + \sum I_k^* \right\} < 1$$

Define $F : PC([0, T_0], X) \rightarrow PC([0, T_0], X)$ by

$$\begin{aligned} Fx(t) = & x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s)x(s)ds \\ & + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0 < t_k < t} I_k x(t_k^-). \end{aligned} \quad (3.2)$$

Thus it can be said that equation (3.1) has unique mild solution if F defined by (3.2) has unique fixed point. This means F is well defined bounded operator on $PC([0, T_0], X)$ and F is contraction [35].

Lemma 3.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \dots, p$ are continuous then F is bounded operator on $PC([0, T_0], X)$.*

Proof. Let a sequence $\{x_n\}$ converges to x in $PC([0, T_0], X)$. Therefore $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Consider,

$$\begin{aligned} \|Fx_n - Fx\|_{PC} &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s)x_n(s) - A(s)x(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s)x_n(s) - A(s)x(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &+ \sum_{0 < t_k < t} \|I_k x_n(t_k^-) - I_k x(t_k^-)\| \end{aligned}$$

Assuming the continuity of A, f, T, S and I_k for $k = 1, 2, \dots, p$ the right side of above expression tends to zero as $n \rightarrow \infty$. Therefore F is continuous on $PC([0, T_0], X)$ and hence F is bounded. □

Theorem 3.2. *If the hypotheses (H1)-(H5) are satisfied, then the fractional impulsive integro-differential equation (1.1) has unique mild solution in $PC([0, T_0], X)$ for $0 < \alpha \leq 1$.*

Proof. To show equation (1.1) has unique mild solution it is sufficient to show F defined (3.2) is contraction. Let x and y in $PC([0, T_0], X)$ and consider,

$$\begin{aligned} \|Fx - Fy\|_{PC} &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s)x(s) - A(s)y(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s)x(s) - A(s)y(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &+ \sum_{0 < t_k < t} \|I_k x(t_k^-) - I_k y(t_k^-)\| \end{aligned}$$

Applying hypotheses (H1)-(H4), it leads to

$$\begin{aligned}
 \|Fx - Fy\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} M \|x - y\| ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} M \|x - y\| ds \\
 &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \{L_1 + T_0 H L_2 + T_0 K L_3\} \|x - y\| ds \\
 &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} ds \{L_1 + T_0 H L_2 + T_0 K L_3\} \|x - y\| ds + \sum_{0 < t_k < t} I_k^* \|x - y\| \\
 &\leq \left\{ \frac{T_0^\alpha}{\Gamma(\alpha + 1)} [(p + 1)[M + L_1 + T_0 H L_2 + T_0 K L_3]] + \sum I_k^* \right\} \|x - y\| \\
 &= \left\{ \gamma [(p + 1)[M + L_1 + T_0 H L_2 + T_0 K L_3]] + \sum I_k^* \right\} \|x - y\|
 \end{aligned}$$

Assuming hypotheses (H5) it follows that, $\|Fx - Fy\|_{PC} \leq q \|x - y\|$ with $q < 1$. Hence by Banach fixed point theorem the equation (1.1) has unique mild solution. \square

3.2 Remarks

- (1) This method suggests not only the existence and uniqueness of mild solution but it also suggests method to find approximate solution of impulsive fractional differential equation (1.1).
- (2) This condition is not necessary condition. This means equation (1.1) may have mild solution if one of the (H1) to (H5) are not satisfied.
- (3) If all I_k 's are constants then assumption (H5) can replace by $q = \left\{ \gamma [(p + 1)[M + L_1 + T_0 H L_2 + T_0 K L_3]] \right\} < 1$

4 Classical Solution

Definition 4.1. A solution $x(t)$ is a classical solution of the equation (1.1) for $0 < \alpha < 1$ if $x(t) \in PC([0, T_0], X) \cap C^\alpha(J', X)$ where, $J' = [0, T_0] - \{t_1, t_2, \dots, t_p\}$, $x(t) \in D(A)$ (Domain of A) for $t \in J'$ and satisfies (1.1) on $[0, T_0]$.

Lemma 4.1. If conditions,

- (B1) $A : X \rightarrow X$ is continuous bounded linear operator and there exists a positive constant M , such that $\|A(t)x - A(t)y\|_{B(X)} \leq M \|x - y\|$, for all $x, y \in X$.
- (B2) $f \in C^\alpha([0, T_0] \times X \times X \times X, X)$ such that, there exists positive constants L_1, L_2 and L_3 , such that $\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq L_1 \|x_1 - y_1\| + L_2 \|x_2 - y_2\| + L_3 \|x_3 - y_3\|$.

(B3) $h : D_0 \times X \rightarrow X$ and $k : D_1 \times X \rightarrow X$ are continuous and there exists positive constants H and K , such that $\|h(t, s, x) - h(t, s, y)\| \leq H\|x - y\|$ and $\|k(t, s, x) - k(t, s, y)\| \leq K\|x - y\|$ for all x and y in X .

(B4) Let, $\gamma = \frac{T_0^\alpha}{\Gamma(\alpha+1)}$ and $q = \left\{ \gamma[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3] \right\} < 1$.

(B5) $x_0 \in D(A)$.

are satisfied then the fractional evolution equation

$$\begin{aligned} {}^c D^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(0) &= x_0 \end{aligned} \quad (4.1)$$

($0 < \alpha \leq 1$) has unique classical solution over $[0, T_0]$ which satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Proof. Applying Riemann- Liouville integral operator both side of equation (4.1) to obtain,

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds. \quad (4.2)$$

It can be easily shown that $x(t)$ defined above is in $C^\alpha([0, T_0], X)$. Therefore to show, $x(t)$ is classical solution, it is sufficient to show that $x(t) \in D(A)$ for all $t \in [0, T_0]$.

Define a sequence $x_n(t)$ by,

$$x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)x_{n-1}(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_{n-1}(s), Tx_{n-1}(s), Sx_{n-1}(s))ds.$$

Clearly for each n , $x_n(t) \in D(A)$ because $x_0 \in D(A)$ and applying assumptions (B1) to (B4) the sequence x_n converges uniformly to x . Hence, $x(t) \in D(A)$ for all t . Therefore, $x(t)$ is classical solution of (4.1) which is of the form (4.2) which is also mild solution of the equation (4.1). Moreover, uniqueness of mild solution gives uniqueness of classical solution. Hence, $x(t)$ is unique classical solution of (4.1) satisfies mild solution (4.2). \square

Next lemma is generalization of the lemma-4.1

Lemma 4.2. *If conditions,*

(C1) $A : X \rightarrow X$ is continuous bounded linear operator and there exists a positive constant M , such that $\|A(t)x - A(t)y\|_{B(X)} \leq M\|x - y\|$, for all $x, y \in X$.

(C2) $f \in C^\alpha([0, T_0] \times X \times X \times X, X)$ such that, there exists positive constants L_1, L_2 and L_3 , such that $\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|$.

(C3) $h : D_0 \times X \rightarrow X$ and $k : D_1 \times X \rightarrow X$ are continuous and there exists positive constants H and K , such that $\|h(t, s, x) - h(t, s, y)\| \leq H\|x - y\|$ and $\|k(t, s, x) - k(t, s, y)\| \leq K\|x - y\|$ for all x and y in X .

(C4) Let, $\gamma = \frac{t_0^\alpha}{\Gamma(\alpha+1)}$ and $q = \left\{ \gamma[(p+1)[M+L_1+t_0HL_2+t_0KL_3] \right\} < 1$ where, $t_0 = (t_k - t_{k-1})$

(C5) $q_{k-1} \in D(A)$.

are satisfied then the fractional evolution equation

$$\begin{aligned} {}^c D_{t_{k-1}^+}^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_{k-1}) &= q_{k-1} \end{aligned} \quad (4.3)$$

($0 < \alpha < 1$) and for $k = 1, 2, \dots, p$ has unique classical solution over interval $[t_{k-1}, t_k)$ which satisfies,

$$x(t) = q_{k-1} + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Moreover, one can define $x(t_k)$ in such a way that, $x(t_k)$ is left continuous and $x(t_k) \in D(A)$.

Proof. Replacing 0 by t_{k-1} and T_0 by t_k and applying lemma-4.1 get, equation (4.3) has unique solution which satisfies

$$x(t) = q_{k-1} + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.. \quad (4.4)$$

Define,

$$x(t_k) = q_{k-1} + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds,$$

then to show, $x(t_k)$ is left continuous consider increasing sequence $\{s_m\}$ converges to t_k i.e., $\|s_m - t_k\| \rightarrow 0$ as $m \rightarrow \infty$. One can easily show that $\|x(s_m) - x(t_k)\| \rightarrow 0$ as $m \rightarrow \infty$. Hence, $x(t)$ is left continuous at t_k and $x(t_k) \in D(A)$. \square

Theorem 4.3. If assumptions (H1) – (H5) is satisfied, $f \in C^\alpha([0, T_0] \times X \times X \times X, X)$ and $x_0 \in D(A)$ then the equation,

$$\begin{aligned} {}^c D^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \quad k = 1, 2, \dots, p \\ \Delta x(t_k) &= r_k, \quad t = t_k, \quad k = 1, 2, \dots, p \\ x(0) &= x_0 \end{aligned} \quad (4.5)$$

($0 < \alpha < 1$) has unique classical solution over the interval $[0, T_0]$ which satisfies

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0 < t_k < t} r_k \end{aligned} \quad (4.6)$$

Proof. Consider the interval $I_1 = [0, t_1]$ then, the equation (4.5) becomes,

$$\begin{aligned} {}^c D^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(0) &= x_0. \end{aligned} \tag{4.7}$$

Using lemma-4.2, equation (4.7) has unique classical solution which satisfies the equation

$$x_1(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \tag{4.8}$$

and define, $x_1(t_1)$ as

$$x_1(t_1) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Also, $x_1(t)$ is left continuous at $t = t_1$ and $x_1(t_1) \in D(A)$.

On the interval $I_2 = [t_1, t_2]$, the equation (4.5) becomes

$$\begin{aligned} {}^c D_{t_1+}^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_1) &= x_1(t_1) + r_1. \end{aligned} \tag{4.9}$$

Since, $x_1(t)+r_k \in D(A)$ therefore applying lemma-4.2 equation (4.9) has unique classical solution which satisfies

$$x_2(t) = [x_1(t_1)+r_1] + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \tag{4.10}$$

and define, $x_2(t_2)$ as

$$x_2(t_2) = [x_1(t_1)+r_1] + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Also, $x_2(t)$ is left continuous at $t = t_2$ and $x_2(t_2) \in D(A)$.

Continuing this process on $I_k = [t_{k-1}, t_k]$, equation (4.5) becomes

$$\begin{aligned} {}^c D_{t_{k-1}+}^\alpha x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_{k-1}) &= x_{k-1}(t_{k-1}) + r_{k-1}. \end{aligned} \tag{4.11}$$

and applying lemma-4.2 equation (4.11) has unique classical solution which satisfies

$$\begin{aligned} x_k(t) &= [x_{k-1}(t_{k-1}) + r_{k-1}] + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t-s)^{\alpha-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t-s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds. \end{aligned} \tag{4.12}$$

and define, $x_k(t_k)$ as

$$x_k(t_k) = [x_{k-1}(t_{k-1}) + r_{k-1}] + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Also, $x_k(t)$ is left continuous at $t = t_k$ and $x_k(t_k) \in D(A)$. Now define

$$x(t) = \begin{cases} x_1(t) & t \in [0, t_1] \\ x_k(t) & t \in [t_{k-1}, t_k] \\ x_{p+1}(t) & t \in [t_p, t_{p+1}] \end{cases} \quad (4.13)$$

This $x(t)$ define by equation (4.13) is unique classical solution of the equation (4.5). Next to show equation (4.13) satisfies (4.6). If $t \in [0, T_0]$ then there exist k such that $t \in I_k = [t_{k-1}, t_k]$. Therefore

$$\begin{aligned} x(t) &= x_k(t) \\ &= [x_{k-1}(t_{k-1}) + r_{k-1}] + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t - s)^{\alpha-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds. \end{aligned}$$

Putting value of $x_{k-1}(t_{k-1}), x_{k-2}(t_{k-2}), \dots, x_1(t_1)$ in above equation and solving to get, equation (4.6). Therefore, under the given assumption equation (4.5) has classical solution which satisfied equation (4.6). \square

Next theorem shows the existence and uniqueness of classical solution which derives from mild solution.

Theorem 4.4. *Assume the hypotheses (H1) – (H5) are satisfied. Let $x(t)$ be mild solution of equation (1.1) obtained in theorem-3.2. Assume that $x_0 \in D(A)$ and $I_k x(t_k) \in D(A)$ for $k = 1, 2, \dots, p$ and $f \in ((0, T_0) \times X \times X \times X, X)$. Then $x(t)$ give rise to unique classical solution of (1.1).*

Proof. Let $x(t)$ be mild solution of (1.1). Replace, $r_k = I_k x(t_k)$ for all $k = 1, 2, \dots, p$. Also given that, (H1) – (H5) satisfied, $x_0, I_k x(t_k) \in D(A)$ for all $k = 1, 2, \dots, p$. Therefore, by theorem-4.3 equation (1.1) has unique classical solution say $y(t)$ which satisfy equation

$$\begin{aligned} y(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s)y(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s)y(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s), Ty(s), Sy(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, y(s), Ty(s), Sy(s))ds + \sum_{0 < t_k < t} I_k y(t_k). \end{aligned}$$

Set, $z(t) = x(t) - y(t)$ then $z(t)$ satisfies evolution equation ${}^c D^\alpha z(t) = 0$ with initial condition $z(0) = 0$ and without impulses. Hence, $z(t) \equiv 0$ is only solution of the evolution equation ${}^c D^\alpha z(t) = 0$. Hence, $x(t) = y(t)$ and therefore $x(t)$ give rise to classical solution of (1.1). \square

Example 4.4.1. Consider an evolution equation

$$\begin{aligned} {}^c D^\alpha x(t) &= \frac{1}{10} \int_0^1 (t-s)x(s)ds + \frac{1}{10} \int_0^t se^{-\frac{x(s)}{4}} ds + \frac{1}{10} \int_0^1 (t-s)^2 e^{x(s)} \\ \Delta x(1/2) &= \frac{1}{10} x(1/2^-) \\ x(0) &= x_0 \end{aligned} \tag{4.14}$$

over the interval $[0, 1]$. Then, $f \equiv \frac{1}{10} \int_0^t se^{-\frac{x(s)}{4}} ds + \frac{1}{10} \int_0^1 (t-s)^2 e^{x(s)}$ with $Tx(s) = \frac{1}{10} \int_0^t se^{-\frac{x(s)}{4}} ds$ and $Sx(s) = \frac{1}{10} \int_0^1 (t-s)^2 e^{x(s)}$. Impulse at $I_1 x(1/2) = \frac{1}{10} x(1/2^-)$. One can easily show that $\|Tx - Ty\| \leq \frac{1}{10} \|x - y\|$, $\|Sx - Sy\| \leq \frac{1}{10}$ and $\|I_1 x(1/2) - I_1 y(1/2)\| \leq \frac{1}{10} \|x(1/2) - y(1/2)\|$. Therefore $\|f(t, x(t), Tx(t), Sx(t)) - f(t, y(t), Ty(t), Sy(t))\| \leq \frac{1}{5} \|x(t) - y(t)\|$. Hence, $q = \left\{ \gamma[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3] + \sum I_k^* \right\} = \frac{1}{\Gamma(\alpha)} \frac{6}{10} < 1$ for any $0 < \alpha \leq 1$. Thus, equation (4.14) has unique mild solution.

Moreover, $f \in C^\alpha((0, 1) \times X \times X \times X, X)$ and choosing, $x_0 \in D(A)$ one get unique classical solution of equation which is arise from mild solution.

5 Discussion

In this article authors modified the work of Aaunguraj and Arjunan [22] first order impulsive evolution equation to fractional order. However authors in [22] used operator semi-group theory to prove the results. But, fractional order system does not have semi-group property so authors of this article used different approach to prove modified results. One can also get existence and uniqueness of mild and classical solutions by weaken Lipschitz type conditions used by authors.

6 Conclusion

Many of the authors discussed existence and uniqueness of mild solutions but only mild solution some times is not sufficient in many of the practice situations and requires classical solution. So there has to be sufficient conditions for classical solutions and conditions in which mild solution becomes classical solution.

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