Analysis of Absolute Stability of Non – Linear Systems Using Lyapunov Method

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Abstract
In this paper we are analyzing the stability of non-linear systems by Lyapunov stability functions. This paper focus on the two methods of Lyapunov stability: Lyapunov indirect method and Lyapunov direct method. concerning the stability of solutions near to a point of equilibrium is the most important type. The idea of system linearization around a given point is used for Lyapunov indirect method and local stability with small stability regions can be achieved. On the other hand the Lyapunov direct method is used for the design and analysis of nonlinear systems. Together , the indirect method and direct method constitute the so- called Lyapunov stability theorem.

Keywords: *Non-linear systems, Lyapunov stability functions, Lyapunov's direct method, Lyapunov's indirect method*

I. Introduction

In practice, the situation in which the transfer of function of a closed loop control system is not exactly known for various reasons (eg: non linearity, difficulty in obtaining an exact dynamic model, unmodelled dynamics etc). In this case, the Lyapunovs method can be applied to test the asymptotical stability of a close loop control system.Aleksandr Mikhailovich Lyapunov was a Russian mathematician, mechanic and physicist. He was born on June 6th,1857. His sur name is sometimes romanized as Ljapunov, Liapunov, Liapounoff, and Ljapunow. He is popularly known for his Lyapunov function, Lyapunov stability, Lyapunov exponent, Lyapunov central limit theorem and Lyapunov vector. M. Lyapunov was a pioneer in fruitful trying to build up the worldwide way to deal with the examination of the security of non direct dynamical frame works y correlation with broadly spread neighbourhood technique for linearizing them about purposes of hormony. His work at first distributed in Russian and afterward meant French got little consideration for a long time .the numerical hypothesis of solidness of moment established by A.M .Lyapunov significantly foreseen the ideal opportunity for its use age in signs and innovation. Additionally Lyapunov did not himself make application in this field ,his own enthusiasm being in the security of pivoting liquid masses with galactic application. He didn't have doctoral understudies who pursued the exploration in the field of solidness and his own predetermination was horrendously sad as a result of Russian unrest of 1917.for a very long while the hypothesis of soundness sank into complete blankness .Lyapunov introduced two methods,

- The first method is called Lyapunov's first or indirect method. It is a linearization technique.
- The second method is Lyapunov's second or direct method. This is a generalization of Lagrange's concept of stability of minimum potential energy.

II. Lyapunov Function

When the scalar function V(x) which is a function of vector x, this function is called Lyapunov function if and only if the following condition holds: And V(x) has a continuous-time derivative (i.e. V(x) exists for all t>0.

In practice, there are many scalar functions which are qualified to be Lyapunovs functions. This is an advantage. For example, if x(t) is a continuous-time vector, V(x)=12(t)*x(t) is a Lyapunov function. In fact, a Lyapunov function is also known as a control objective function because the objective of control action is to make this function converge to zero.

Advantages

- In the case there is Lyapunov function, this is a generalization of the minimal energy principle, to study the stability near equilibrium, even for nonlinear ordinary differential equations.
- Asymptotic stability can be checked using Lyapunov function when exists. Sometimes global stability is checked using this process.
- Lyapunov depend ability of a hormony point in space is the most crucial idea about soundness of arrangements of differential conditions in view of the thought other variation sorts of depend ability can be determined for example.

- Lyapunov stability helps in analyzing the stability of nonlinear systems.

Disadvantages

- There is no systematic way of obtaining Lyapunov functions.
- Lyapunov stability criterion provides only sufficient condition for stability.

Stability

Stability is the state or quality of being steady and not changing. There are two types of stability:

- Absolute stability
- Relative stability

Absolute Stability

Absolute stability means whether the system is stable or unstable.
Example: A numerical method is said to be absolutely stable if all of its roots lies within the unit circle.

Relative Stability

Relative security gives the level of depend ability or that it is so near unsteaadyness.
Example: If we consider two systems, one system is towards zero and another system lies before zero, then we can say that the system two is more stable as compared to the system one, then that system is known as relatively stable in context to root locus.

III. Lyapunov's Direct Method

Basic Theorem of Lyapunov

Table 1. Condition on V(x,t) and Condition on $\dot{V}(x,t)$

Let V(x,t) be a non-negative function with derivative along the trajectories of the system.

1. If V(x,t) is locally positive definite and $\dot{V}(x,t) \leq 0$ locally in x andfor all t, then the origin of the system is locally stable (in the sense of Lyapunov).

2. If V(x,t) is locally positive definite and decrescent, and $\dot{V}(x,t) \leq 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).
3. If V(x,t) is locally positive definite and decrescent, and $-\dot{V}(x,t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.

S.No	Condition on V(x,t)	Condition on $-\dot{V}(x,t)$	Conclusions
1	Lpdf	locally	Stable
2	lpdf, decrescent	locally	Uniformly stable
3	lpdf, decrescent	Lpdf	Uniformly asymptotically stable
4	pdf, decrescent	Pdf	Globally uniformly asymptotically stable

4. If V(x,t) is positive definite and decrescent, and $\dot{V}(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.

Proof: In general, proving that a nonlinear system of the form
$\dot{x}(t) = f(x(t))$
(1)
is asymptotically stablearound the origin is a difficult task, as it is difficult to write down a closed form solution of x(t)
in terms of x(t0). For linear time-invariant systems
$(\dot{x}(t) = Ax(t) + Bu(t))$,
(2)
we have a closed formsolution, i.e.,

$$x(t) = e^{A(t-t_0)}x(t0) + \int_{t0}^{t} e^{A(t-T)}Bu(T)dT \quad (3)$$

For any A (regardless of whether it is diagonalizable or not), the linear system $\dot{x} = Ax$ is asymptotically stable at the origin if and only if all the Eigen values of A lie in the open left-half complexplane1. This is due to the decaying exponential terms in x(t). On the other hand, if the system
has at least one Eigen value in the open right-half plane, then the linear system is unstable (i.e. not Stable i.s.L.). Stability of the system (i.s.L.) if it has Eigen values on the imaginary axis depends onthe algebraic and geometric multiplicity of these Eigenvalues. We assume that the origin is the only equilibrium point, i.e., A has full rank to analyze the stability of general nonlinear systems, we can carefully choose a scalar-valuedfunction,
V (x), of the state variables and see how V (x) evolves as the states evolve.

Example 1: In this example, we will prove global stability of the equilibrium point using thedirect method.

$\dot{x}_1 = -x_1 - x_2$
(4)
$\dot{x}_2 = x_1 - x^3_2$
(5)

First, note that the origin is the only equilibrium point of the system. Furthermore, linearizingabout the origin, we obtain the dynamics of the linearized system,

$$\begin{bmatrix}\Delta\dot{x}1\\ \Delta\dot{x}2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta\dot{x}1\\ \Delta\dot{x}2\end{bmatrix}$$
(6)

The Eigen values of the linearization are given by $-0.5 \pm j0.5\sqrt{3}$, so we may conclude, via Lyapunov'sindirect method, that the origin is locally asymptotically stable. However, the indirect method doesnot provide any information about how close to the origin we have to be to guarantee stability.
Thus, we turn to the Lyapunov's direct method.
Consider using a candidate Lyapunov function V (x) of the form $ax_1^2+bx_2^2$where a and b are somepositive parameters to be determined. Clearly V (x) is positive definite over the entire state spaceand V (x) is radially unbounded. Now, we need to check that $\dot{V}(x)$ is negative definite:

$$\dot{V}(x) = [2ax1 \quad 2bx2]\begin{bmatrix} -x1 & -x2 \\ x1 & -x2^3 \end{bmatrix} \quad (7)$$

$$= -2ax_1^2-2x_2^4+2x_1x_2(b-A) \quad (8)$$

If we choose a = b = 1/2, then$\dot{V}(x)=-x_1^2-x_2^4< 0$ for all $x \neq 0$.
Thus, this system is G.A.S.

Example 2: In this example, we find that the origin is locally asymptotically stable. We cannot Conclude global asymptotic stability of the origin due to the presence of more than one equilibrium point; but, using Lyapunov's direct method, we can find the set of initial conditions for which thesystem trajectories will eventually end up at the origin (this ball is called the region of attraction).Consider the autonomous (time-invariant) system described by the differential equations:

$$\dot{x}_1=(x_1-x_2)(x_1^2+x_2^2-1) \quad (9)$$
$$\dot{x}_2=(x_1+x_2)(x_1^2+x_2^2-1) \quad (10)$$

This system has an infinite number of equilibrium points: one at the origin and the rest on theunit circle. Since there are multiple equilibria, none of the equilibria can be globally asymptoticallystable; furthermore, since the points on the unit circle are not isolated, none can be locally asymptotically stable. We wish to examine the equilibrium point at the origin. Linearizing the systemabout the origin, we find that that the linearized dynamics have Eigenvalues at $-1 \pm j$, so we canconclude, via the indirect method, local asymptotic stability (L.A.S.) of the origin.

Now, using the direct method, we can find the set of initial conditions for which the system trajectories converge to 0. Consider $V(x) = ax_1^2+ bx_2^2$, with a, b >0, which is positive definite for all x_1 and x_2(in R^2). Now,

$$\dot{V}(x) =[2ax_1 \quad 2x_2]\begin{bmatrix}(x_1-x_2)(x_1{}^2+x_2{}^2-1) \\ (x_1+x_2)(x_1{}^2+x_2{}^2-1)\end{bmatrix} \quad (11)$$

$$= (2ax_1^2+2x_1x_2(b-a))(x_1^2+x_2^2-1) \quad (12)$$

Note that $x_1^2-x_2^2-1 < 0$ when $\|x\|_2<1$, $x\neq 0$. If b = a = 1/2, then
$\dot{V}(x)=(x_1^2+x_2^2)(x_1^2+x_2^2-1)<0$
for all nonzero x in the open unit circle. Thus the origin is L.A.S. within the region $\|x\|_2< 1$.

Example 3: Consider the following differential equations for a∈R using the direct method.

$x_1(t) = x_2(t)$
(13)
$x_2(t) = -x_1(t)+\alpha x_1^2(t)x_2 t$
(14)

Lyapunov function,
$V(x_1,x_2) = \frac{1}{2}(x_1^2+x_2^2)$
(15)

$V(t) = x_1(t)\overline{x_1}(t)+x_2(t)\overline{x_2}(t)$
(16)

$= x_1(t)x_2(t)+x_2(t)(-x_1(t))+\alpha x_1^2(t)x_2(t)$

$=\alpha x_1^2(t)\, x_2^2(t)$
(17)

If $\alpha \leq 0$, V is negative, then the equilibrium
$x^*=(0,0)$ is stable
If $\alpha<0$, V is negative , then the equilibrium is
$x^*=(0,0)$ is asymptotically stable.

Example 4: Consider the following differential equations for a∈R using the direct method.

$\frac{\boldsymbol{dx}}{\boldsymbol{dt}}=y - 2x$, $\frac{dy}{dt}=2x - y - x^3$
(18)

Lyapunov function,
$V(x) = V(x,y)$
(19)
$=(x+y)^2+\frac{x^4}{2}$
(20)

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} + \frac{dv}{dy}\frac{dy}{dt}$$

$=(2x+2y+2x^3)(y-2x)+(2x+2y)(2x-y-x^3)$

$=(2x+2y)\,(y-2x)+2x^3(y-2x)-\,(2x+2y)\,(y-2x)-x^3(2x+2y)$

$=2x^3y-4x^4-2x^4-2x^2y$

$= -bx^4 \leq 0$
(21)

Since the derivative is negative at (0,0) , then the solution is asymptotically stable.

IV. The Indirect Method of Lyapunov Stability

Instead of looking for a Lyapunov function to be applied directly to the nonlinear system, one can linearize the system around the origin and attempt to conclude "local" stability of the origin using quadratic Lyapunov functions for the linearized system. More specifically, if the linearized system's A matrix has Eigen values in the open LHP (Left-Hand Path), the nonlinear system is then locally asymptotically stable (L.A.S.).
Consider the system

$$\dot{x}=f(x,t) \quad \text{(V.1)}$$

With f(0,t) = 0 for all t ≥ 0. Define

$$A(t)=\partial f(x,t)/\partial x|_{x=0} \quad \text{(V.2)}$$

To be the jacobian framework of f(x,t) as for x as if that the starting point it pursues that or each fixed t the rest of

$$f_1(x,t)=f(x,t)-A(t)x \quad \text{(V.3)}$$

approaches zero as x approaches zero. However, the remainder may not approach zero uniformly. For this to be true, the stronger condition is required, that is

$$\lim_{\|x\|\to 0} \sup_{t \geq 0} \frac{\|f1(x,t)\|}{\|x\|} = 0 \quad \text{(V.4)}$$

If the above equation holds, then the system

$$\dot{Z}=A(t)z$$

is referred to as the (uniform) linearization of equationabout the origin. When the linearization exists, its stability determines the local stability of the original non-linear equation.

THEOREM OF STABILITY BY LINEARIZATION
Consider the system (V.1) and assume

$$\lim_{\|x\|\to 0} \sup_{t \geq 0} \frac{\|f1(x,t)\|}{\|x\|} = 0$$

Further, let A(•) defined in equation (V.2) be bounded. If 0 is a uniformly asymptotically stable equilibrium point of (V.4) then it is a locally uniformly asymptotically stable equilibrium point of (V.1).

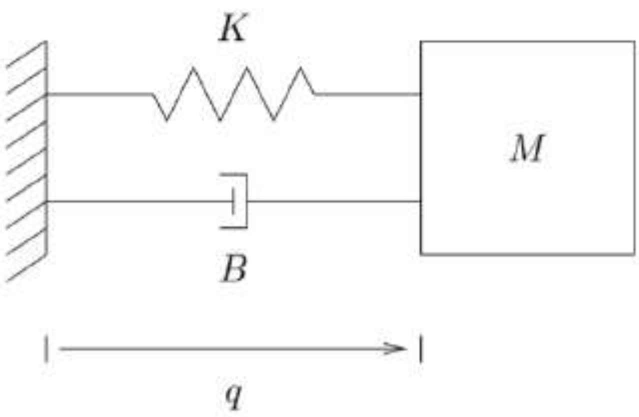

Figure 1.Damped harmonic oscillator.

The formal hypothesis requires uniform asymptotic depend ability of the linearlized framework to demonstrate uniform asymptotic strength of the nonlinear framework. Counter examples to the theorem exist if the linearized system is not uniformly asymptotically stable.
If the system (V.1) is time-invariant, then the indirect method says that if the Eigen values of

$A=\frac{\partial f(x)}{\partial x}|x=0$

Are in the open left half complex plane at that point the sources asymptotically steady .this hypothesis demonstrates that worldwide uniform asymptotic depend ability of the linearization suggest nearby uniform asymptotic strength of the first nonlinear frame work. The evaluation gaveby the evidence of the hypothesis can be utilized to give a bound on the area of fascination of the starting point .Systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of non-linear systems is an important area of research and involves searching for the "best" Lyapunov functions.

Example 1: consider the autonomous pendulum with friction

$x_1= -x_2$
(22)
$x_2=x_1+(x_1^2-1)x_2$
(23)

where a,b>0. The Jacobian matrix A at the equilibrium point x=0 is given by:

$A= \partial f/\partial x|_{x=0}=\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}$
(24)

Then the eigenvalues of A are $(-1\pm j\sqrt{3})/2$. Hence , the origin is asymptotically stable. Taking Q=I the Lyapunov equation becomes

$PA+A^TP= -I$
(25)

$P=\begin{bmatrix}1.5 & -0.5\\ -0.5 & 1\end{bmatrix}$
(26)

And $\lambda_{min}(P)=0.691$. thus, the system has a candidate Lyapunov function

ISSN: 2005-4238 IJAS

$V(x)=x^TPx$
(27)

$=1.5x_1^2-x_1x_2+x_2^2$
(28)

We obtain the derivative of V(x) as

$V(x)=(3x_1-x_2)(-x_2)+(-x_1+2x_2)[(x_1+(x_1^2-1)x_2)]$
(29)

$= -(x_1^2+x_2^2)-(x_1^3x_2 -2x_1^2x_2^2)$

$\leq -\|x^2\|+|x_1|\|x_1x_2\||x_1-2x_2|$

$\leq -\|x\|^2+\frac{\sqrt{5}}{2}\|x\|^4$
(30)

where $|x_1| \leq \|x_1\|, |x_1x_2| \leq \frac{1}{2}\|x\|^2, |x_1-2x_2| \leq \sqrt{5}\|x\|$. Thus , we obtain

$V(x)<0,\ 0<\|x\|^2<\frac{2}{\sqrt{5}}$
(31)

Obviously , taking $c=\lambda_{min}(P)r^2=0.619\times\frac{2}{\sqrt{5}}=0.618$, {V(x)<c} is an estimate of the region of attraction.

V. Conclusion

In this paper a review of studying of stability of nonlinear control systems using the Lyapunov methods has been proposed, for practical implementation we have to find the Lyapunov function explicitly for each nonlinear framework ,our proposal is that the Lyapunov work normally speaks to the complete vitality of the framework. Its derivative is continuously decreasing so that the system eventually reaches an equilibrium point and mains at that point. The usefulness of both Lyapunov indirect and direct method has been presented . Furthermore we have referred to conditions for absolute stability and relative stability , and the use of Lyapunov function for determination of a domain of attraction . models have been proposed to outline the methods for examining the solidness of nonlinear framework.

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ISSN: 2005-4238 IJAS