International Journal of Advanced Science and Technology
Vol. 29, No. 12s, (2020), pp. 2330-2341

Double Domination Number of Subgroup Lattice of a Group

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Abstract

A subgroup lattice $\sum(Z_n)$ is a diagram that includes all the subgroups of the group Z_n and then if H and K are subgroups of Z_n with $H \subsetneq K$ and there is no subgroup J such that $H \subsetneq J \subsetneq K$, then K appear above H and a segment is drawn connecting H and K. The graph of subgroups lattice is denoted by $\sum(Z_n)$. A subset D^d of $V(\sum Z_n)$ is double dominating set of $\sum(Z_n)$ if for every vertex $< [v] > \in V(\sum(Z_n))$, $|N[< [v] >] \cap D^d| = 2$, that is $< [v] >$ in D^d and has at least one neighbour in D^d or $< [v] > \in V(\sum(Z_n) - D^d$ and has at least two neighbours in D^d. The double domination number $\gamma_{dd}(\sum(Z_p))$ in subgroup lattice $\sum(Z_n)$ is a minimum cardinality of double dominating set. In this paper some upper and sharp bounds on $\gamma_{dd}(\sum(Z_p))$ are obtained

Kay Words: *Double Dominating set, Double Domination, Subgroup lattice.*

Mathematics Subject Classification 2010: 05C25, 05C15, 13E15.

1. Introduction

All graphs considered here are simple that is finite, undirected and loopless. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The open neighbourhood $N(v)$ of the vertex v consists of vertices adjacent to v, $N(v) = \{u \in V: (u, v) \in E\}$ and the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. The concept of subgroup lattice of a graph was introduced in [1]. A subgroup lattice provides a visual depiction of the subgroup structure of a group. A subgroup lattice is a diagram that includes all the subgroups of the group and if H and K are subgroups of G with $H \subsetneq K$ and there is no subgroup J such that $H \subsetneq J \subsetneq K$, then K appear above H and a segment is drawn connecting H and K. The graph of subgroup lattice in denoted by $\sum(Z_n)$. A subset D^d of $V(G)$ is a double dominating set of G if for every vertex $v \in V(G), |N[v] \cap D^d| \geq 2$, that is v is in D^d and has at leat one neighbor in D^d or v is in $V(G) - D^d$ and has at least two neighbors in D^d. In this paper I give some upper and sharp bounds of double domination number of subgroup lattice of a group. For a survey on the area of double domination in graphs I refer the reader to [1, 5, 6, 7, 8, 9, 10, 11, 12].

2. Preliminary Definitions

Some preliminary definitions of graph theory and algebra for more details the reader is referred to [2, 3, 4].

Definition: A group $\langle G, *\rangle$ is a set G, closed under a binary operation $*$, such that following axioms are stisfied:

(i) **Closure axiom:** $\forall\, a, b\ \in G \Longrightarrow a * b \in G.$

(ii) **Associativity law:** $(a * b) * c = a * (b * c)\ \forall a, b, c \in G.$

(iii) **Existence of identity:** $\exists$ an element $e \in G$, called identity such that $a * e = e * a = a\ \forall\ a \in G.$

(iii) **Existence of Inverse:** Corresponding to very $a \in G, \exists\ a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = a$. This a^{-1} is called inverse of a.

Definition: Let $\langle G, *\rangle$ be a group. A non-empty subset H of G is said to be a subgroup of G if $\langle H, *\rangle$ is itself a group we write this as $H \leq G$ and read that H is a subgroup of G.

Definition 2.3: The cycle $C_n, n \geq 3$, consist of n vertices $v_1, v_2, \dots, v_n$ and edges $\{v_1,\ v_2\}$, $\{v_2,\ v_3\},\ \dots, \{v_n, v_1\}$.

Definition 2.4: Deleting an edge from a cycle graph C_n a path graph of order n is obtained and a path graph of order n is denoted by P_n.

Definition 2.5: The ladder graph L_n is defined by $L_n = P_n \times K_2$ where P_n is path with n vertices and $\times$ denotes the Cartesian product and K_2 is a complete graph with two vertices.

Definition 2.6: The complement or inverse of a graph G is a graph H on the same vertices such that distinct vertices of H are adjacent if and only if they are not adjacent in G.

3. SUBGROUP LATTICE OF A GROUP AND SOME EXAMPLES

In this Section we observe some examples.

Examples: Consider Z_n, the group of integers modulo n.

(i) Let us construct the subgroup lattice of group Z_n, where $n = p$, p is prime number.

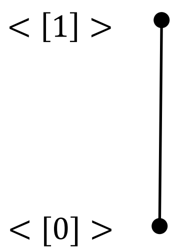


Figure 3.1(i) $\sum(Z_p)$

Theorem 3.1: For the subgroup lattice $\sum(Z_p)$, where p is prime, $\gamma_{dd}\left(\sum(Z_p)\right) = 2$.

Proof: Let $V(\sum(Z_p)) = \{< [0] >, < [1] >\}$ be the set of vertices of $\sum(Z_p)$. Let $< [0] >$ and $< [1] >$ be two vertices of a graph $\sum(Z_n)$ and they are adjacent if and only if $< [0] > \subsetneq < [1] >$. $D^d = \{< [1] >, < [0] >\}$ be the minimal double dominating set. Since each and every vertex of D^d has one neighbour in D^d, therefore D^d is the double dominating set with double domination number 2. Hence $\gamma_{dd}\left(\sum(Z_p)\right) = 2$.

(ii) Let us construct the subgroup lattice of group Z_n, where $n = p^2$.

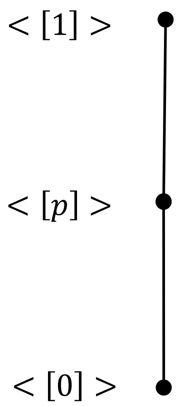


Figure 3.2(ii) $\sum(Z_{p^2})$

Theorem 3.2: For the subgroup lattice $\sum(Z_{p^2})$, where p is prime, $\gamma_{dd}\left(\sum(Z_{p^2})\right) = 3$.

Proof: Let $V(\sum(Z_p)) = \{< [0] >, < [p] >, < [1] >\}$ be the set of vertices and $E(\sum(Z_n)) = \{(< [0] >, < [p] >), (< [p] >, < [1] >\}$ be the set of edges of $\sum(Z_{p^2})$. Let $D^d = \{< [0] >, < [p] >, < [1] >\}$ be the minimal double domination set. Since each and every vertex of D^d has one neighbourhood in D^d and there is no single vertex other than other than vertices of D^d, therefore D^d is the minimal double set with the domination number 3. Hence $\gamma_{dd}(\sum(Z_p)) = 3$.

(iii) Let us construct the subgroup lattice of group Z_n, where $n = p^3$. Now $\sum(Z_{p^3})$ is given in

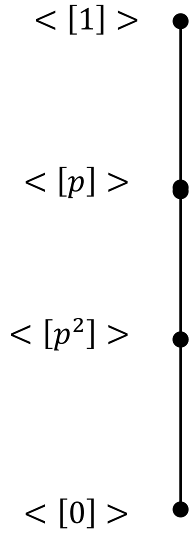
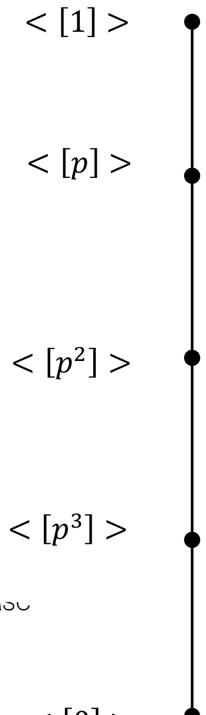


Figure 3.3(iii) $\sum(Z_{p^3})$

Theorem 3.3: For the subgroup lattice $\sum(Z_{p^3})$, where p is prime, $\gamma_{dd}\left(\sum\left(Z_{p^3}\right)\right) = 4$.

Proof: Let $V(\sum(Z_{p^3})) = \{ < [0] >, < [p^2] >, < [p] >, < [1] > \}$ be the set of vertices of $\sum(Z_{p^3})$. Let $V_1\left(\sum\left(Z_{p^3}\right)\right) = \{< [1] >, < [p^2] >\}$ and $V_2\left(\sum\left(Z_{p^3}\right)\right) = \{< [p] >, < [0] >\}$ be the two independent set of vertices of $\sum(Z_{p^3})$. Let $D^d = V_1\left(\sum\left(Z_{p^3}\right)\right) \cup V_2\left(\sum\left(Z_{p^3}\right)\right)$ be the double dominating set of $\sum(Z_{p^3})$ such that $|N[< [v] >] \cap D^d| \geq 1 \ \forall < [v] > \in D^d$. Thus $|D^d| = \left|V_1\left(\sum\left(Z_{p^3}\right)\right) \cup V_2\left(\sum\left(Z_{p^3}\right)\right)\right|$. Hence $\gamma_{dd}(\sum(Z_{p^3})) = 4$.

(iv) Let us construct the subgroup lattice of group Z_n, where $n = p^4$.



ISSN: 2005-4238

Figure 3.4(iv) $\sum(Z_{p^4})$

Theorem 3.4: For the subgroup lattice $\sum(Z_{p^4})$, where p is prime, $\gamma_{dd}\left(\sum\left(Z_{p^4}\right)\right) = 4$.

Proof: Let $V\left(\sum(Z_{p^4})\right) = \{<[1]>, <[p]>, <[p^2]>, <[p^3]>, <[0]>\}$, $E\left(\sum(Z_{p^4})\right) = \{(<[0]>, <[p^3]>), (<[p^3]>, <[p^2]>), (<[p^2], <[p]>), (<[p]>, <[1]>)\}$ be the set of vertices and edges of $\sum(Z_{p^4})$. Let $D^d = \{<[1]>, <[p]>, <[p^3]>, <[0]>\}$ be the minimal double dominating set of $\sum(Z_{p^4})$ such that the vertex $<[p^2]> \in V\left(\sum(Z_{p^4})\right) - D^d$ is adjacent to $<[p]>$ and $<[p^3]>$ of D^d and $|N<[p^2]> \cap D^d| = 2$. Hence $\gamma_{dd}\left(\sum(Z_{p^4})\right) = 4$.

(v) Let us construct the subgroup lattice of group Z_n, where $n = p^5$.

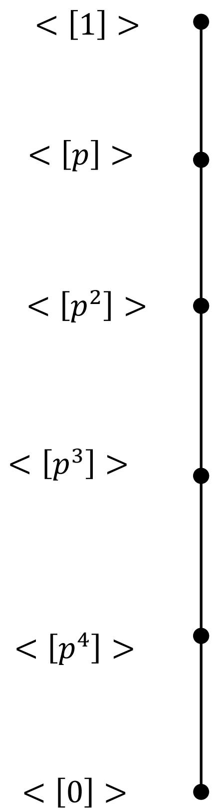


Figure 3.5 (v) $\sum(Z_{p^5})$

Theorem 3.5: For the subgroup lattice $\sum(Z_{p^5})$, where p is prime, $\gamma_{dd}\left(\sum(Z_{p^5})\right) = 5$.

Proof: Let $V\left(\sum(Z_{p^5})\right) = \{<[1]>, <[p]>, <[p^2]>, <[p^3]>, <[p^4]>, <[0]>\}$ be the set of vertices of $\sum(Z_{p^5})$. Let $V\left(\sum(Z_{p^5})\right) = V_1\left(\sum(Z_{p^5})\right) = \{<[1]>, <[p^2]>, <[p^4]>\} \cup \left(\sum(Z_{p^5})\right) = \{<[p]>, <[p^3]>, <[0]>\}$. Let $D^d = \{<[1]>, <[p]>, <[p^2]>, <[p^4]>, <[0]>\} = V_1\left(\sum(Z_{p^5})\right) \cup V_2\left(\sum(Z_{p^5})\right) - \{<[p^3]>\}$ be the minimal double dominating set of $\sum(Z_{p^5})$ such that $|N <[p^3]> \cap D^d| = 2$. Clearly$|D^d| = \left|V_1\left(\sum(Z_{p^5})\right)\right| \cup \left|V_2\left(\sum(Z_{p^5})\right)\right| - 1$. Hence $\chi\left(\sum(Z_{p^5})\right) = 5$.

Observation 1: For the subgroup lattice $\sum(Z_{p^n})$, where p is prime, $\gamma_{dd}\left(\sum(Z_{p^n})\right) \leq n$.

(vi) Let us construct the subgroup lattice of group Z_n, where $n = pq$, where p and q are distinct odd prime numbers and $p < q$.

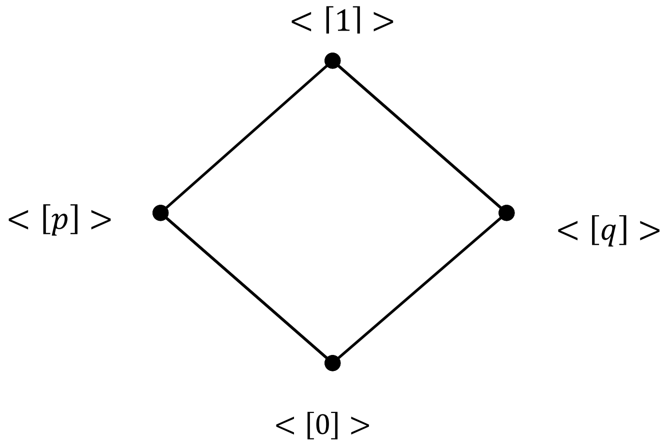


Figure 3.6(vi) $\sum(Z_{pq})$

Theorem 3.6: For the subgroup lattice $\sum(Z_{pq})$, where p and q are distinct odd prime numbers and $p < q$, $\gamma_{dd}\left(\sum(Z_{pq})\right) = 3$.

Proof: Let $V\left(\sum(Z_{pq})\right) = \{<[1]>, <[p]>, <[q]>, <[0]>\}$ and $E\left(\sum(Z_{pq})\right) = \{(<[0]>, <[p]>), (<[0]>, <[q]>), (<[p]>, <[1]>), (<[q]>, <[1]>)\}$ be the set of vertices and edges of $\sum(Z_{pq})$. Let $D^d = \{<[p]>, <[q]>, <[0]>\}$ be the minimal double dominating set such that the vertex $<[1]> \in V\left(\sum(Z_{pq})\right) - D^d$ has two neighbour in D^d that is $|N[<[1]>] \cap D^d| = 2$. Hence $\gamma_{dd}\left(\sum(Z_{pq})\right) = 3$.

(vii) Let us construct the subgroup lattice of a group Z_n, where $n = p^2q$, p and q are distinct prime.

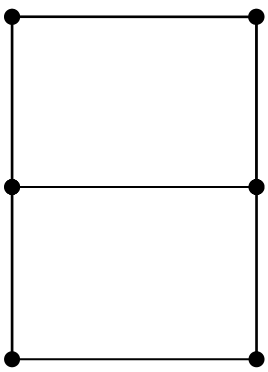


Figure 3.7 (vii) $\sum(Z_{p^2q})$

Theorem 3.7: For the subgroup lattice $\sum(Z_{p^2q})$, where p and q are distinct prime, $\gamma_{dd}(\sum(Z_{p^2q}) = 4$.

Proof: Let $V(\sum(Z_{p^2q}) = \{<[1]>, <[p]>, <[p^2]>, <[q]>, <[pq]>, <[0]>\}$ be the set of vertices of and $E(\sum(Z_{p^2q}) = \{(<[u]>, <[v]>): <[u]> \subsetneq <[v]>\}$ be the set of edges of $\sum(Z_{p^2q})$ where $<[u]>$ and $<[v]>$ belong to $V(\sum(Z_{p^2q})$. Let $= V_1(\sum(Z_{p^2q})) = \{<[1]>, <[p^2]>\}$ and $V_2(\sum(Z_{p^2q})) = \{<[q]>, <[0]>\}$ be the set of vertices of $\sum(Z_{p^2q})$. Let $D^d = \{<[1]>, <[p^2]>, <[q]>, <[0]>\} = V_1(\sum(Z_{p^2q})) \cup V_1(\sum(Z_{p^2q}))$ be the minimal double dominating set of $\sum(Z_{p^2q})$ such that any vertex $<[v]> \in V(\sum(Z_{p^2q}) - D^d$ has two neighbours in D^d and $|N[<\{v\}>] \cap D^d| = 2$. Therefore D^d is the double dominating set with double domination number 4. Hence $\gamma_{dd}(\sum(Z_{p^2q}) = 4$.

(viii) Let us we construct subgroup lattice of a group Z_n, where $n = 2p$, p is prime and $p > 2$.

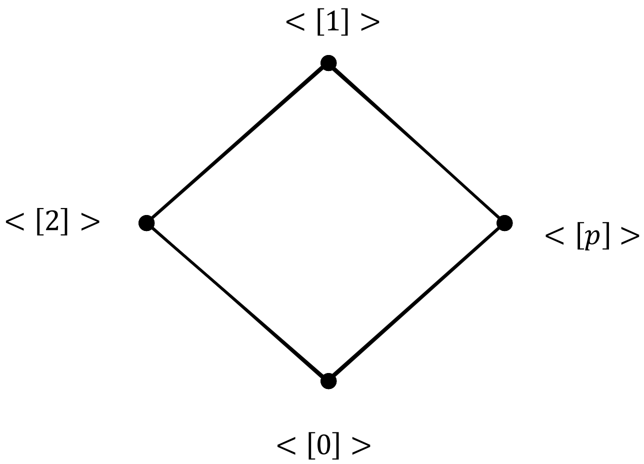


Figure 3.8 (viii) $\sum(Z_{2p})$

Theorem 3.8: For the subgroup lattice $\sum(Z_{2p})$, where p is prime and $p > 2$, $\gamma_{dd}(\sum(Z_{2p})) = 3$.

Proof: The result follows from theorem 3.6.

(ix) Let us we construct the subgroup lattice for a graph Z_n where $n = 3p$, p is prime and $p > 3$

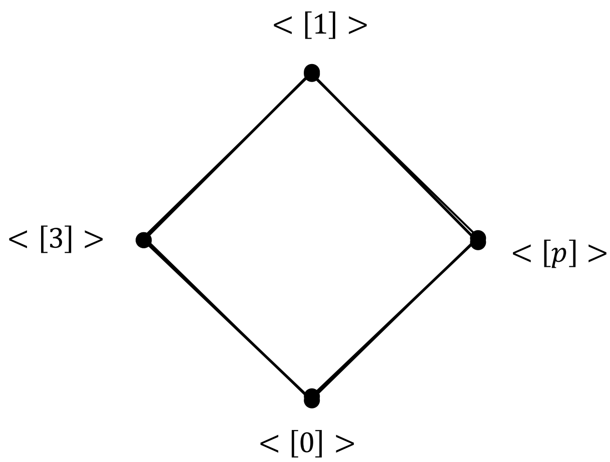


Figure 3.9 (ix) $\sum(Z_{3p})$

Theorem 3.9: For the subgroup lattice $\sum(Z_{3p})$, where p is prime and $p > 3$, $\gamma_{dd}\left(\sum(Z_{3p})\right) = 3$.

Proof: The result follows from theorem 3.6.

(x) Let us construct the subgroup lattice for a graph Z_n, where $n = 4p$, p is prime and $p > 4$.

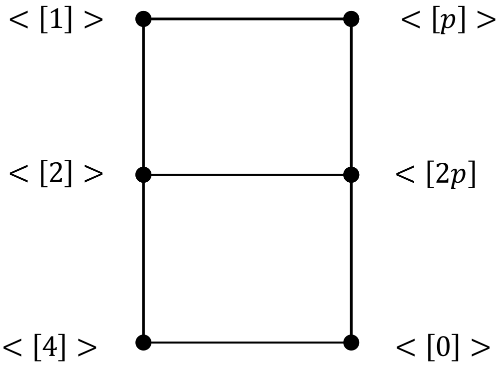


Figure 3.10 (x) $\sum(Z_{4p})$

Theorem 3.10: For the subgroup lattice $\sum(Z_{4p})$, where p is prime and $p > 5$, $\gamma_{dd}\left(\sum(Z_{4p})\right) = 4$.

Proof: The result follows from theorem 3.7.

(xi) Let us construct the subgroup lattice for the graph Z_n, where $n = 8p$, p is prime and $p > 8$.

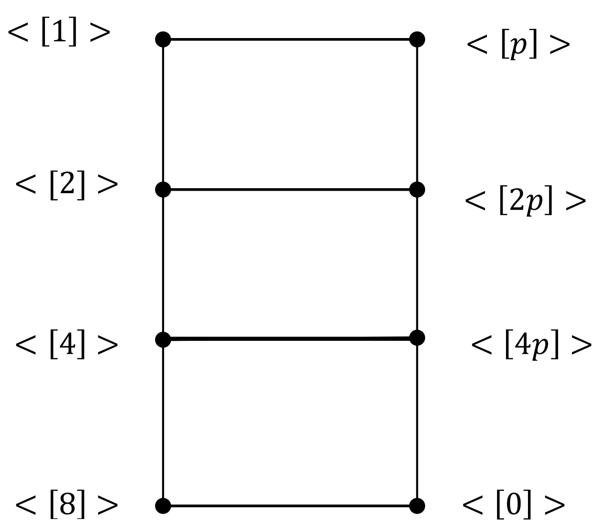


Figure 3.11 (xi) $\sum(Z_{8p})$

Theorem 3.11: For the subgroup lattice $\sum(Z_{8p})$, where p is prime and $p > 8$, $\gamma_{dd}\left(\sum(Z_{8p})\right) = 5$.

Proof: Let $V(\sum\left(Z_{p^2q}\right) = \{<[1]>, <[2]>, <[4]>, <[8]>, <[p]>, <[2p]>, <[2q]>, <[4p]>, <[0]>\}$ be the set of vertices and $E(\sum\left(Z_{p^2q}\right) = \{(<[u]>, <[v]>): <[u]> \subsetneq <[v]>\}$ be the set of edges of $\sum(Z_{p^2q})$ where $<[u]>$ and $<[v]>$ belong to $V(\sum\left(Z_{p^2q}\right)$. Let $= V_1\left(\sum\left(Z_{p^2q}\right)\right) = \{<[1]>, <[4]>, <[8]>\}$ and $V_2\left(\sum\left(Z_{p^2q}\right)\right) = \{<[p]>, <[4p]>\}$ be the set of vertices of $\sum(Z_{p^2q})$. Let $D^d = \{<[1]>, <[4]>, <[8]>, <[p]>, <[4p]>\} = V_1(\sum(Z_{p^2q})) \cup V_1(\sum(Z_{p^2q}))$ be the minimal double dominating set of $\sum(Z_{8p})$ such that any vertex $<[v]> \in V(\sum\left(Z_{p^2q}\right) - D^d$ has two neighbours in D^d and $|N[<\{v]>] \cap D^d| = 2$. It is clear that $|D^d| = \left|V_1(\sum(Z_{p^2q})) \cup V_1(\sum(Z_{p^2q}))\right| = 5$. Hence $\gamma_{dd}\left(\sum(Z_{8p})\right) = 5$.

(xi) Let us construct the subgroup lattice for a group Z_p, where $p = 2^n$, $n > 2$.

a) Let $n = 3$, $\sum(Z_{2^3})$ b) Let $n = 4$, $\sum(Z_{2^4})$ c) Let $n = 5$, $\sum(Z_{2^5})$

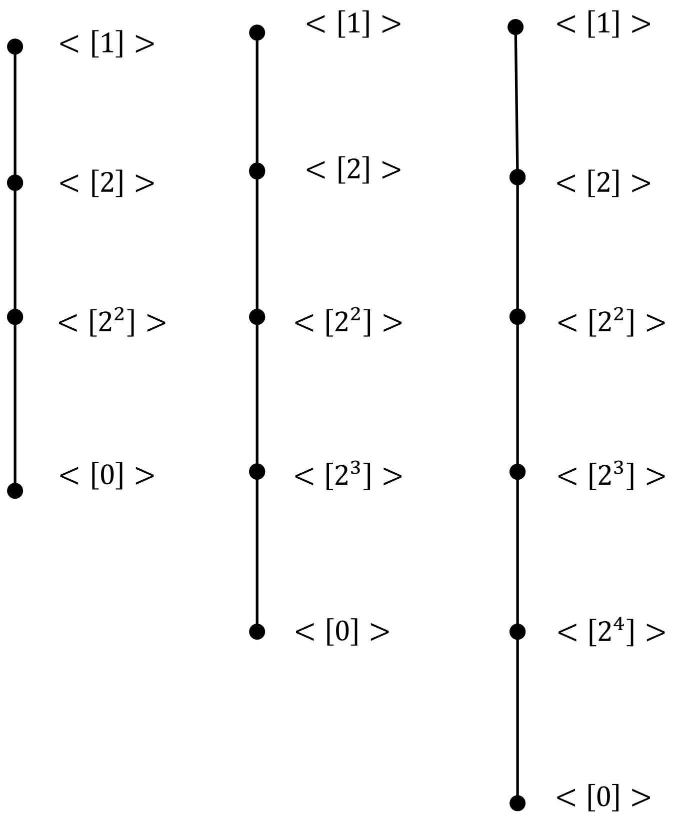


Figure 3.12 (xii) $\sum(Z_{2^n})$

Theorem 3.12: For the subgroup lattice $\sum(Z_{2^n})$, where $n > 2$, $\gamma_{dd}(\sum(Z_{2^n})) \leq n + 1$.

Proof: By applying theorem 3.3, theorem 3.4 and theorem 3.5 we get the result.

(xiii) Let us we construct the subgroup lattice for the group Z_p, where $p = 3^n$, $n > 3$. a) Let $n = 4$, $\sum(Z_{3^4})$ b) Let $n = 5$, $\sum(Z_{3^5})$, c) Let $n = 6$, $\sum(Z_{3^6})$

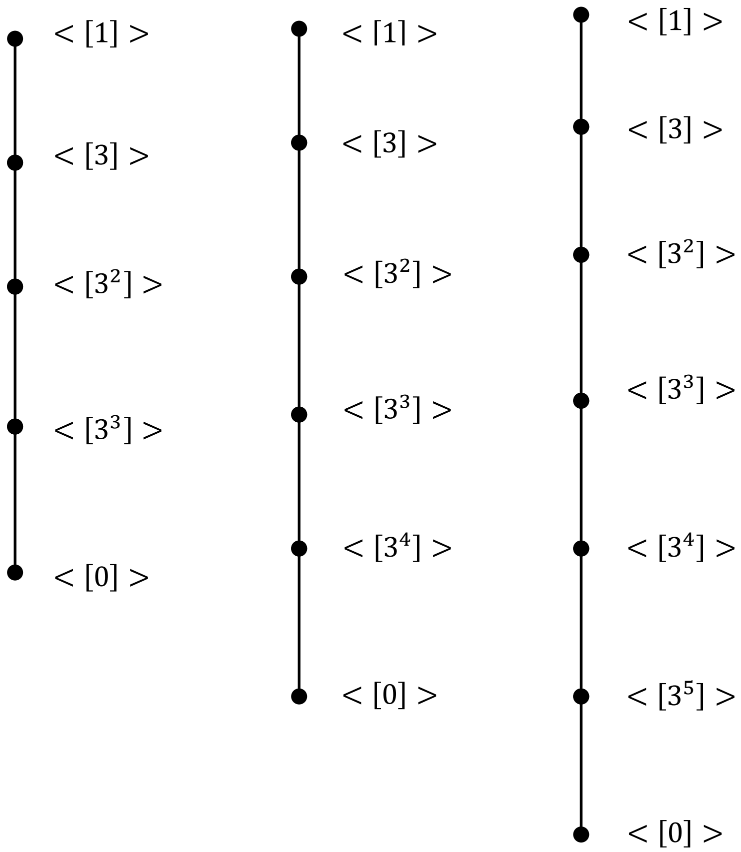


Figure 3.13 (xiii) $\sum(Z_{3^n})$

Theorem 3.13: For the subgroup lattice $\sum(Z_{3^n})$, where $n > 3$, $\gamma_{dd}(\sum(Z_{3^n})) \leq n$.

Proof: By applying theorem 3.3, theorem 3.4 and theorem 3.5 we get the result

4. NORDHAUS-GADDUM TYPE RESULT

Theorem 4.1: For any subgroup lattice $\sum(Z_n)$ with $t =< [v] > \geq 2$ vertices

1) $\gamma_{dd}(\sum(Z_n)) + \gamma_{dd}(\sum(\bar{Z}_n)) \leq 2t$

2) $\gamma_{dd}(\sum(Z_n))\gamma_{dd}(\sum(\bar{Z}_n)) \leq t^2$.

CONCLUSION

Double Domination is a particular type of domination and the double domination in subgroup lattice $\sum(Z_n)$ is relative new research area of domination theory. In this paper some upper and sharp bounds on $\gamma_{dd}(\sum(Z_p))$ are obtained and Nordhaus-Gaddum type result are also obtained.

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