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Cordiality Of Edge Graceful Cayley Digraphs

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Abstract
Here we go with the definition of Q(a) P(b) super edge graceful labelling (SEG) and examine the above labelling on the Cayley digraph. We investigate the digraph with E_k cordiality also.. We derive a characterisation theorem relating E_k cordial with Edge graceful.
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1. INTRODUCTION

Rosa introduced β-valuation for a graph G [7]. S. W. Golomb[5] later named the labelings as Graceful and it is the common phrase at present. Volumes of papers have derived parallels for graceful graphs by revising node and edge labels. S. Lo [6] presented the concept of edge-graceful labelling for graphs. We have shown odd regular digraphs are edge graceful [8]. Subsequently many authors developed edge graceful labeling methods with some changes [4]. One such labeling method is Q(a) P(b) super edge graceful labeling and introduced by Chopra and Lee [3]. We extended the same for digraphs and prove it for Cayley digraphs.

Cordial labeling is derived from two significant labelings graceful and harmonious. The theory of cordiality was presented by Cahit.I [2]. In [9] Yilmaz and Cahit discussed the new concept called E_k cordial labeling for undirected Graphs. Afterwards, Bapat and Limaye discussed the same for complete graphs [1]. In this paper we prove it for Cayley digraphs.

2. PRELIMINARIES

Definition 2.1: Consider two positive integers d and b. A (p, q) digraph is Q(d) P(b) super edge graceful or simply(d, b) –SEG if it does exist an onto mapping f: E(G)→ Q(d) and $f^*: V(G) \to P(b)$, where

$$Q(d) = \{\pm d, \pm(d+1), \dots, \pm\left(d+\frac{q-2}{2}\right)\} \text{ if q is even,}$$

$$= \{0, \pm d, \pm(d+1), \dots, \pm\left(d+\frac{q-3}{2}\right)\} \text{ if q is odd,}$$

$$P(b) = \left\{\pm b, \pm(b+1), \dots, \pm\left(b+\frac{p-2}{2}\right)\right\} \text{ if p is even,}$$

$$= \left\{0, \pm b, \pm(b+1), \dots, \pm\left(b+\frac{p-3}{2}\right)\right\} \text{ if p is odd,}$$

such that $f^*(v) = \sum f(vu)$ here the total is taken over the weights of the outgoing arcs of the node v. Moreover the digraph is said to be strongly SEG if the digraph satisfies Q(d) P(b) SEG for all a ≥ 1.

Definition 2.2: A digraph is called E_k cordial if it's outgoing arcs of every vertex are labelled from the set $\{0, 1, 2, \dots, k-1\}$ so that the sum of the labels of outgoing arcs from every node v under modulo k meets the inequalities $|v(i) - v(j)| \le 1$ and $|e(i) - e(j)| \le 1$, where v(s) and e(t) are the number of nodes labeled with s and the number of arcs labeled with t respectively.

3. MAIN RESULTS

Here we examine that the existence of Q(d) P(b) super edge graceful and E_k cordial on Cayley digraph.

Theorem 3.1.1

The digraph of Cayley group admits Q(d) P(b) super edge graceful labelling (SEG).

Proof:

Let the digraph having n nodes, m generators, and then the Cayley digraph has totally mn arcs. Now Mark the node set of the digraph as V={v_1, v_2, …,v_n} .
The arc set of Cay(G,S)as E= E_{s1} UE_{s2}U…UE_{sm} = {e_{11}, e_{12}, …,e_{1m},$e_{21}$$e_{22}$…$e_{2m}$ ... e_{n1} …e_{nm}} here
E_{sm} = The set containing arcs produced by s_m outgoing from every v_i.
Where e_{ij} is j^{th} arc produced by s_j going out from vertex v_i

To prove Cayley digraph admits Q(d) P(b) SEG, we have to prove the following. For any two positive integers a and b, there exist an onto mapping

f: E(G)→ Q(d) and $f^*: V(G) \to P(b)$, where

$$Q(d) = \{\pm d, \pm(d+1), \dots, \pm\left(d+\frac{q-2}{2}\right)\} \text{ if q is even,}$$

$$= \{0, \pm d, \pm(d+1), \dots, \pm\left(d+\frac{q-3}{2}\right)\} \text{ if q is odd,}$$

$$P(b) = \left\{\pm b, \pm(b+1), \dots, \pm\left(b+\frac{p-2}{2}\right)\right\} \text{ if p is even,}$$

$$= \left\{0, \pm b, \pm(b+1), \dots, \pm\left(b+\frac{p-3}{2}\right)\right\} \text{ if p is odd,}$$

such that $f^*(v) = \sum f(vu)$ here the sum is over the labels of the arcs outgoing from vertex v. We investigate this statement in four different cases.

(i) m and n are even

If m and n are even then q is also even. Then for any d ≥ 1,

Define f : E(G) → $\{\pm d, \pm(d+1), \dots, \pm\left(d+\frac{q-2}{2}\right)\}$ as

$$f(e_{rj}) = \begin{cases} d+\frac{n(j-1)}{2}+r-1 & if\ j\ is\ odd \\ -[d+\frac{n(j-2)}{2}+r-1] & if\ j\ is\ even\ and\ j \neq m \end{cases}$$

For j = m

$$f(e_{im}) = \begin{cases} -\left[d+\frac{n(m-2)}{2}+2(r-1)+1\right] & for\ 1 \le r \le \frac{n}{2} \\ -[d+\frac{n(m-2)}{2}+2\left(r-1-\frac{n}{2}\right)] & for \frac{n}{2}+1 \le r \le n \end{cases}$$

Then the induced function

$$f^*(v_i) = \sum_{j=1}^{m} f(e_{rj})$$

For $1 \leq r \leq \frac{n}{2}$,

$$f^*(v_i) = d + i - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-1-1)}{2} + r - 1 - (a + \frac{n(m-2)}{2} + 2(r-1) + 1)$$
$$= -r \qquad \ldots\ldots(1)$$

For $\frac{n}{2} + 1 \leq r \leq n$,

$$f^*(v_r) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-1-1)}{2} + r - 1 - [d + \frac{n(m-2)}{2} + 2(r-1) - n]$$

$$= n + 1 - i\mathrm{r} \qquad \ldots\ldots(2)$$

From equation (1) we have $-\frac{n}{2} \leq f^*(v_r) \leq -1$ and from (2), we have $1 \leq f^*(v_r) \leq \frac{n}{2}$

So, for $1 \leq r \leq n$ and for any $d \geq 1$, $f^*(v_r) = \left\{\pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\}$ = P(b) where b =1.
Therefore Cayley digraph is Q(d) P(b) SE graceful when the number of vertices and generators are even.

(ii) m is even and n is odd

Since we took m as even, q is also even.
Then for any $d \geq 1$,

Define f : E(G) → $\{\pm d, \pm(d+1), \ldots, \pm\left(d + \frac{q-2}{2}\right)\}$ as

$$f(e_{ij}) = \begin{cases} d + \frac{n(j-1)}{2} + r - 1 & if\ j\ is\ odd \\ -[d + \frac{n(j-2)}{2} + r - 1] & if\ j\ is\ even\ and\ j \neq m \end{cases}$$

For j = m

$$f(e_{im}) = \begin{cases} -\left[d + \frac{n(m-2)}{2} + 2(r-1) + 1\right] & for\ 1 \leq r \leq \frac{n-1}{2} \\ -[d + \frac{n(m-2)}{2} + 2\left(r - 1 - \frac{n-1}{2}\right)] & for\ \frac{n-1}{2} + 1 \leq r \leq n \end{cases}$$

Then the resulting function

$$f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})$$

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For $1 \leq r \leq \frac{n-1}{2}$,

$$f^*(v_r) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-1-1)}{2} + r - 1 - (d + \frac{n(m-2)}{2} + 2(r-1) + 1)$$

$$= -r \qquad \ldots\ldots(3)$$

For $\frac{n-1}{2} + 1 \leq r \leq n$,

$$f^*(v_r) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-1-1)}{2} + r - 1 - [d + \frac{n(m-2)}{2} + 2(r-1) - (n-1)]$$

$$= n - r \qquad \ldots\ldots(4)$$

From equation (3) we have $-\frac{n-1}{2} \leq f^*(v_r) \leq -1$ and from (4), we have $1 \leq f^*(v_r) \leq \frac{n-1}{2}$. Also when i = n, $f^*(v_i) = 0$. So, for $1 \leq i \leq n$ and for any a ≥ 1,$f^*(v_r) = \left\{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\right\}$ = P(b) where b=1. Therefore Cayley digraph of odd number of nodes and even number of generators is Q(d) P(b) SEG.

(iii) m is odd and n is even

Since n is even, q = mn is also even.
Then for any a ≥ 1,

Define f : E(G) → $\{\pm d, \pm(d+1), \ldots, \pm\left(d + \frac{q-2}{2}\right)\}$ as

$$f(e_{ij}) = \begin{cases} d + \frac{n(j-1)}{2} + r - 1 & if\ j\ is\ odd\ and\ j \neq m \\ -[d + \frac{n(j-2)}{2} + r - 1] & if\ j\ is\ even \end{cases}$$

For j = m

$$f(e_{im}) = \begin{cases} d + \frac{n(m-1)}{2} + r - 1 & for\ 1 \leq r \leq \frac{n}{2} \\ -[d + \frac{n(m-1)}{2} + r - \frac{n}{2} - 1] & for \frac{n}{2} + 1 \leq r \leq n \end{cases}$$

Then the induced function

$$f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})$$

For $1 \leq r \leq \frac{n}{2}$,

$$f^*(v\ \ ) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-2-1)}{2} + r - 1 - \left(d + \frac{n(m-1-2)}{2} + r - 1\right) + d + \frac{n(m-1)}{2} + r - 1$$

$= d + \frac{n(m-1)}{2} + r - 1$ ……(5)

For $\frac{n}{2} + 1 \leq r \leq n$, 9941959461

$$f^*(v_i) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-2-1)}{2} + r - 1 - \left(d + \frac{n(m-1-2)}{2} + r - 1\right) - [d + \frac{n(m-1)}{2} + r - 1 - \frac{n}{2}]$$

$= -[d + \frac{n(m-1)}{2} + r - 1 - \frac{n}{2}]$ ……(6)

From equation (5) we have $d + \frac{n(m-1)}{2} \leq f^*(v_r) \leq d + \frac{n(m-1)}{2} + \frac{n}{2} - 1$ and from (6), we have $-\left[d + \frac{n(m-1)}{2} + \frac{n}{2} - 1\right] \leq f^*(v_r) \leq -[d + \frac{n(m-1)}{2}]$.So, for $1 \leq r \leq n$ and for any $d \geq 1$,
$f^*(v_r) = \left\{\pm\left[d + \frac{n(m-1)}{2}\right], \pm\left[d + \frac{n(m-1)}{2} + 1\right], \ldots, \pm\left[d + \frac{n(m-1)}{2} + \frac{n}{2} - 1\right]\right\}$ = P(b)
where b=$d + \frac{n(m-1)}{2}$. Therefore Cayley digraph of even number of vertices and odd number of generators is Q(d) P(b) super edge graceful.

(iv) m and n are odd

Since both m and n are odd, q is also odd.
Then for any $d \geq 1$,

Define f : E(G) → $\{0, \pm d, \pm(d+1), \ldots, \pm\left(d + \frac{q-2}{2}\right)\}$ as

$$f(e_{ij}) = \begin{cases} d + \frac{n(j-1)}{2} + r - 1 & if\ j\ is\ odd\ and\ j \neq m \\ -\left[d + \frac{n(j-2)}{2} + r - 1\right] & if\ j\ is\ even \end{cases}$$

For j = m

$$f(e_{im}) = \begin{cases} d + \frac{n(m-1)}{2} + r - 1 & for\ 1 \leq r \leq \frac{n-1}{2} \\ -\left[d + \frac{n(m-1)}{2} + r - 1 - \frac{n-1}{2}\right] & for \frac{n-1}{2} + 1 \leq r \leq n - 1 \\ 0 & for\ r = n \end{cases}$$

Then the resulting function

$$f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})$$

For $1 \leq r \leq \frac{n-1}{2}$,

$$f^*(v_r) = a + r - 1 - (a + r - 1) + a + n + r - 1 - (a + n + r - 1) + \cdots + a + \frac{n(m-2-1)}{2} + r - 1 - \left(a + \frac{n(m-1-2)}{2} + r - 1\right) + a + \frac{n(m-1)}{2} + r - 1$$

$= a + \frac{n(m-1)}{2} + r - 1$ ……(7)

For $\frac{n-1}{2} + 1 \leq r \leq n - 1$,

$$f^*(v_r) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-2-1)}{2} + r - 1 - \left(d + \frac{n(m-1-2)}{2} + r - 1\right) - [d + \frac{n(m-1)}{2} + r - 1 - \frac{n-1}{2}]$$

$$= -[d + \frac{n(m-1)}{2} + r - 1 - \frac{n-1}{2}] \quad \ldots\ldots(8)$$

For r = n,

$$f^*(v_r) = d + r - 1 - (d + r - 1) + d + n + r - 1 - (d + n + r - 1) + \cdots + d + \frac{n(m-2-1)}{2} + r - 1 - \left(d + \frac{n(m-1-2)}{2} + r - 1\right) + 0$$

$$= 0 \quad \ldots\ldots(9)$$

From equation (7) we have $d + \frac{n(m-1)}{2} \leq f^*(v_r) \leq d + \frac{n(m-1)}{2} + \frac{n-1}{2} - 1$ from (8), we have $-\left[d + \frac{n(m-1)}{2} + \frac{n-1}{2} - 1\right] \leq f^*(v_r) \leq -[d + \frac{n(m-1)}{2}]$ and from (9), $f^*(v_r) = 0$

So, for $1 \leq r \leq n$ and for any $a \geq 1$,
$f^*(v_i) = \left\{0, \pm\left[d + \frac{n(m-1)}{2}\right], \pm\left[d + \frac{n(m-1)}{2} + 1\right], \ldots, \pm\left[d + \frac{n(m-1)}{2} + \frac{n-1}{2} - 1\right]\right\}$ = P(b)
where b=$d + \frac{n(m-1)}{2}$. Therefore Cayley digraph is Q(d) P(b) SEG when the number of vertices and the number of generators are odd.

Hence the Cayley digraph is Q(a) P(b) SEG.

Corollary3.1.2

From theorem 3.1.1,we found that Cayley digraph admits Q(a) P(b) SEG labeling for any a ≥ 1 in all the four cases. So The digraph of Cayley is strongly SEG also.

3.2 E_k Cordial labeling of Cayley digraph

Theorem 3.2.1

The digraph of Cayley group G of order n with generating set S is E_k cordial.

Proof:

Construct the Cayley digraph as in the statement and let e_{ij} is j^{th} arc going out of node v_i produced by s_j. To prove the Cayley digraph Cay (G,S) is E_k cordial, we assign labels of outgoing arcs of every vertex from the set $\{0, 1, 2, \ldots, k-1\}$ so that the sum of the labels of outgoing arcs from every vertex v under modulo k meets the inequalities $|v(i) - v(j)| \leq 1$ and $|e(i) - e(j)| \leq 1$, where v(s) and e(t) denote the number of nodes labeled with s and the number of arcs labeled with t respectively. We examine this statement in three cases depending on the number of vertices.

Case (i): k = n

Here we take k is equal to the number of vertices and m is number of generators.
Define f : E → $\{0,1,2, \ldots, k-1\}$ as
f(e_{ij}) ≡ $[r + j - 2](mod\ k)$ where $1 \leq r \leq n\ and\ 1 \leq j \leq m$…..(1)

Then the induced function, $f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})\ (mod\ k)$

$= r - 1 + r + r + 1 + \cdots + r + m - 2 = mr - 1 + 1 + 2 + \cdots + m - 2$

$$= \frac{1}{2}[2mr - 3m + m^2] \equiv \frac{m}{2}[2r - 3 + m](mod\ k) \text{ …..(2)}$$

Since k = n, we get different values of $f^*(v_r)$ under modulo k for every r.
Therefore v(s) = v(t) = 1 for all s and t. $|v(s) - v(t)| = 0 \leq 1$ for all s & t. Since k = n and $1 \leq r \leq n$, for any $s \neq t$, we have e(s) = e(t) = m for all s & t under modulo k [From equation (1)].
$|e(s) - e(t)| = 0 \leq 1$ for all s & t.
Hence the cayley digraph Cay (G,S) is E_k cordial if k = n.

Case (ii): if k < n

Here we take k is less than the number of vertices.
Define f : E → $\{0,1,2, \ldots, k-1\}$ as
f (e_{rj}) ≡ $[r + j - 2](mod\ k)$
Then the resulting function, $f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})(mod\ k)$

$$= \frac{1}{2}[2mr - 3m + m^2] \equiv \frac{m}{2}[2r - 3 + m](mod\ k)$$

Since k < n, we have v(s) = v(t) or v(s) = v(t) + 1 for any $s \neq t$. Therefore $|v(s) - v(t)| \leq 1$ for every s & t.
For any $s \neq t$, we have e(s) = e(t) or e(s) = e(t) + 1 for all s & t. Therefore $|e(s) - e(t)| \leq 1$ for any s & t.

The cayley digraph Cay (G,S) is E_k cordial, if k < n.

Case (iii): if k > n

Now we take k is greater than the number of vertices
Define f : E → $\{0,1,2, \ldots, k-1\}$ as

f (e_{ij}) ≡ $[(j - 1)n + i - 1](mod\ k)$

Then the resulting function, $f^*(v_r) = \sum_{j=1}^{m} f(e_{rj})(mod\ k)$

$$= r - 1 + n + r - 1 + 2n + r - 1 + \cdots + (m - 1)n + r - 1$$

$$= mr - m + n(1 + 2 + \cdots + m - 1) \equiv \frac{m}{2}[2r - 2 + n(m - 1)](mod\ k)$$

Since k > n, v(s) = v(t) =1 for any $s \neq t$. Therefore $|v(s) - v(t)| \leq 1$ for every s & t.
For any $s \neq t$, we have e(s) = e(t) or e(s) = e(t) + 1 for any s & t. Therefore $|e(s) - e(t)| \leq 1$ for any s & t. Hence the Cayley digraph with generating set S is E_k cordial.

Observation3.2.2 If G(n,q) is a Cayley digraph having the property $n = mp$, here m denotes the number of generators and $m \equiv 1(mod\ 2)$, then an edge graceful graph with edges labeled 1 to q is same as labelling 1,2,…,n exactly m times so that induced vertex labeling gives distinct labels under modulo n. we can summarize the above observation in following theorem

Theorem 3.2.3

If G(n, q) is a Cayley digraph with odd generators, then the digraph G is edge graceful if and only if G is E_k cordial for any $k \geq n$.

4. CONCLUSION AND FUTURE WORK

In this article, we exhibited that the digraph of Cayley group G admits Q(a) P(b) SEG labeling and also we have shown that Cayley digraph is strongly super edge graceful and E_k cordial. Also, we have derived a characterization between edge graceful and E_k cordial labeling of Cayley digraphs. In future, we have planned to apply these labelings on other digraphs also.

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